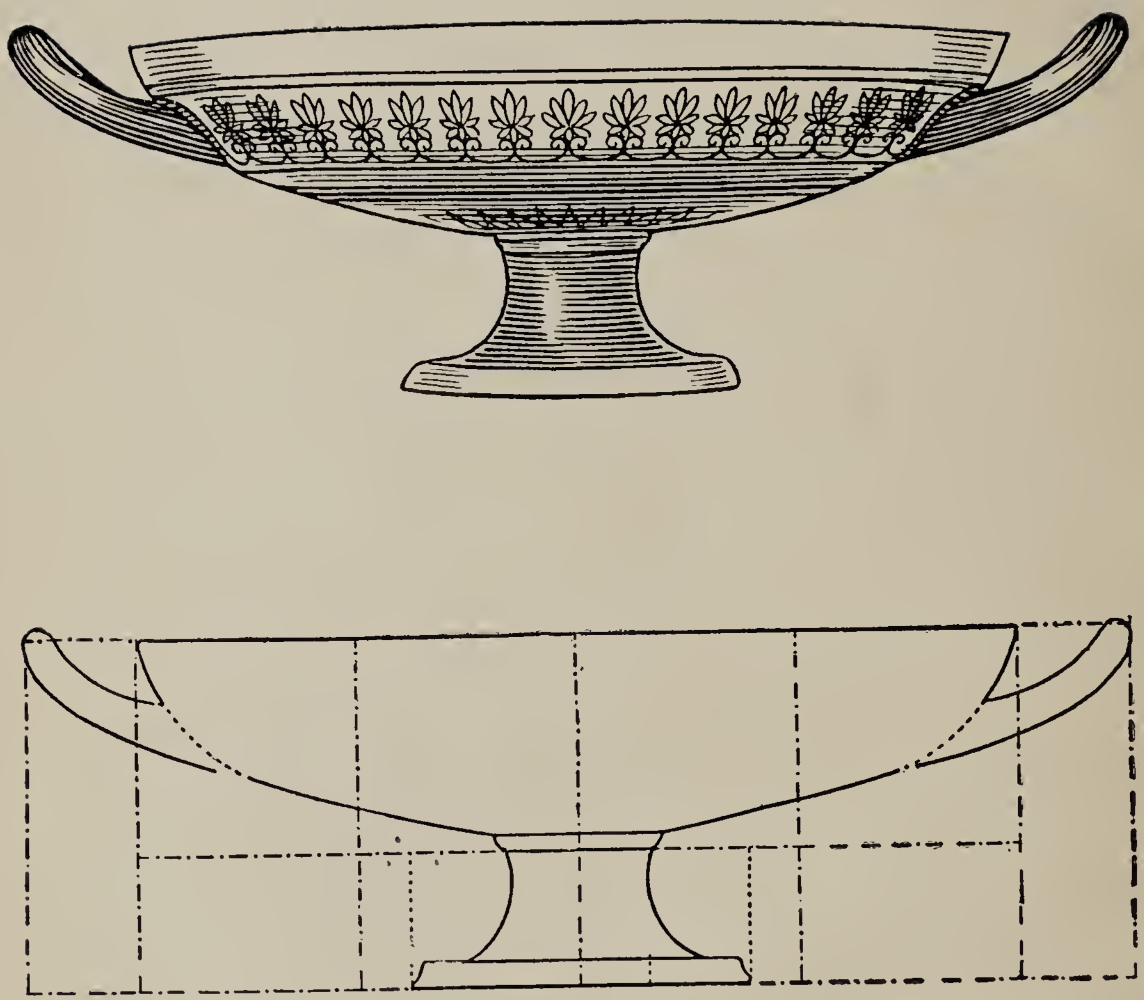
JUNIOR MATHEMATICS



A Greek Vase and the Scale Drawing upon which the Design is Based

JUNIOR MATHEMATICS

BOOK ONE

BY

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AUTHOR'S PREFACE

The study of certain phases of mathematics deserves an important place in a liberal education. Modern thought and enterprise are steadily increasing in mathematical precision, as is apparent in the statistical aspects of the biological and human-nature sciences. To understand modern civilization in many of its aspects one must be able to appreciate its precise quantitative character. One must be able to read intelligently quantitative accounts of modern enterprises. Familiarity with modern methods of measurement and skill in doing quantitative thinking are essential. This text offers a *general* course intended for those who continue studies in the senior high school as well as for those who do not continue. It aims to contribute to the pupil's liberal education by preparing him to understand the quantitative aspects of contemporary civilization.

Since this is a first course in secondary school mathematics, it presupposes knowledge of the fundamental operations with whole numbers, with common fractions, and with decimals. As soon as proficiency in these operations has been attained, the pupil is prepared to take up the study of this course.

The following aims have been set up in the selection and organization of material:

1. The material used must fill a real need in the life of the pupil. It must be useful to him in his present

school studies as well as in preparing him to perform activities in later life. It must have social value.

In this textbook the social realities of the material are emphasized throughout. Free use is made of discussions, pictures, and diagrams to make the social worth of the material appeal to the pupil.

2. The material must be adapted to the abilities of pupils of the early adolescent period and lie within their experience.

Since the success of a course must depend largely on the extent to which this adaptation is accomplished, all the material has been tried out with junior high school classes. On the basis of results of carefully made tests, subject matter not adapted to the mental ability of these pupils has been rejected or changed in treatment. Thus, business applications have been limited to such matters as pupils may be expected to appreciate and understand. Ideas of percentage, interest, etc., which everybody should know, are presented concretely. This is considered sufficient for the majority of pupils. If in a school there are enough pupils intending to enter commercial work, they may be given an additional vocational course in commercial arithmetic. It is advised that vocational courses be given near the end of the junior high school course, or in the senior high school, rather than in the first year.

3. It is not sufficient to teach mathematics merely as a body of principles. There ought also to result training in mathematical methods of thought, effective habits of study as applied to mathematical situations, a conviction of the universal applicability of powers of concentration, and insight into the method of sound

generalization in any field. Such larger values cannot be depended upon to come of themselves. Their achievement has been a matter of constant attention throughout the course.

4. Quantitative relations are to be studied in three ways: geometrically, as in length, area, and graphs; algebraically, as in formulas, equations, and functions; and arithmetically, as in tables and evaluation.

Geometry in its simplest form, because of its usefulness and concreteness, has been made the core of the course. It is experience getting in space relations. It is intuitional, experimental, constructional, not demonstrative. The principles established are those that appeal to the pupil as valuable information. This experience in geometry makes it possible to make the beginning of topics in algebra concrete and then to pass from the concrete to the abstract.

Algebra is not taught in this text as an organized science, but as a helpful tool in the study of other topics. Formulas and equations are the outcome of concrete problems and relate to real things. Since the geometry is concerned only with the conception of plane figures, such as line-segments, angles, and areas, no algebraic functions of degree higher than the second are introduced. These functions are to be considered in the second and third course where three-dimensional figures are introduced and where algebra is studied as a science.

An abundance of work in arithmetic has been provided. The arithmetic aims to secure proficiency in the fundamental manipulative processes needed in the course of ordinary life and in the acquisition of further

mathematical knowledge. Arithmetic is reviewed mainly through applications found within the domain of child life, through wide experiences in many situations. Thus, the arithmetic has changed from the formal drill process used in the lower grades, to a process of assimilation through application.

The following are some of the important features of this text:

The material is organized in pedagogical units, rather than in logical units, *i.e.*, material has been put together which is most economically and effectively learned together. Instead of learning a number of isolated facts or lessons, the pupil sees the relations of a compact body of facts closely related to one another and to the major topic. He will therefore not only master the unit with economy of effort, but will retain it more permanently than when facts are studied separately.

The method of approach is inductive. This is the method of the beginner. The main object is thorough understanding of the concepts to be derived from numerous instances familiar to the pupil by means of examining, contrasting, and comparing. Formal statements, defining the new term or stating the principle are always the last step in the development.

New terms are introduced when they are needed, and not earlier. Moreover, invariably such explanations are given in the text as are necessary to let both the teacher and the pupil understand the reason for bringing in new terms. Each new topic is started with a real problem impressing the pupil with the usefulness of the subject to be studied.

The book contains a great many type examples, illustrating both the method and the arrangement of a solution. Neat and clean-cut written work in mathematics is an important factor in clean-cut thinking. Hence this feature is stressed throughout.

The language of the book is precise, but the sentences are simple so that the pupil can read and understand them. Problems are arranged in order of difficulty determined from actual pupil-performance. There is not a problem in the book which has not been worked by pupils.

In the process of finding the solutions of a problem and in the interpretation of the solutions, it is important that the pupil understand the meaning of approximated measures. Lack of appreciation of the degree of precision causes not only misleading impressions, but also useless effort and waste of time. Since measurement plays an important part in this course and since many data are necessarily approximations, the pupil must learn to determine the degree of precision in the results determined from these data. Hence, considerable attention is given to this matter.

The author wishes to express his appreciation to Director Chas. H. Judd and to Professor H. C. Morrison and the late Professor S. C. Parker of the School of Education, the University of Chicago, for work on the manuscript and for their constant advice and inspiration during the process of development of this course, and to Professor W. C. Reavis whose interest and support as principal of the University High School has greatly facilitated experimentation in junior high school classes.

The studies which contributed to the refinement of the material used in the course presented in this book were aided by a grant from the Commonwealth Fund.

E. R. Breslich.

CONTENTS

INTRODUCTION

The junior high school is recognized by its most ardent advocates as an immature institution. It has grown up during the last decade in all parts of the United States with astonishing rapidity. It came into being because there was a general demand for more productive teaching in the later years of the elementary school and a better administration of the social and personal lives of children in their early adolescent years. It came into being before suitable materials of instruction were at hand and indeed before those who recognized the need for a new institution were altogether certain what the characteristics of this new institution should be.

There are a few instances on record where the junior high school has been tried and abandoned because those who inaugurated it did not know what to put into the courses of study or how to relate this unit to the established units of the school system. There are numerous other instances where the junior high school has been only moderately successful because it was not adequately organized and equipped to fulfill its special functions.

As experience has accumulated, it has become increasingly evident that the ultimate success of the junior high school depends in a very large measure on the preparation of a new kind of teaching material.

This material must be made up of the fundamentals of high school subjects so arranged that they will gather up and review all of the results of elementary education and at the same time open the way to adult life either in the practical world or through higher and more completely differentiated studies of the senior high school. It is not enough that high school courses be carried back into the earlier years of the school's work, nor is it a solution of the junior high school's problem to mix, without genuine intellectual co-ordination, some of the materials which have heretofore been taught in the seventh and eighth grades with a few of the exercises which used to be given in the ninth grade or higher in the school curriculum. There must be a new and completely integrated body of instructional material capable of bridging the gap which has up to this time separated the elementary grades from later intellectual life.

The field of mathematics was one of the first in which teachers began to experiment in the effort to work out a true combination of elementary and higher materials. Numerous books have appeared which more or less successfully accomplished the purpose of introducing pupils to the fundamentals and applications of all the mathematical sciences.

Professor Breslich has a number of advantages in entering this field of experimentation. His books on combination mathematics for the senior high schools are far and away the most successful books of that type in the English language. They have been extensively used in the ninth year and upper years of the high school. Furthermore, Professor Breslich has had the

opportunity for the last six years of working with a seventh year class in the laboratory school of the University of Chicago where, through actual experimentation with a succession of classes which he has himself taught, he has refined his methods and materials to the point where they can now be offered to a wider constituency.

Two years ago, through a subsidy from the Commonwealth Fund, Professor Breslich was enabled to visit the leading junior high schools of the country and to make a study of their courses in mathematics. He has also conducted courses in the principles of teaching mathematics in the School of Education of the University of Chicago, and has in this way come into contact with experienced teachers and supervisors from all parts of the country.

This book is accordingly one of the maturest courses for junior high schools that has been prepared. It follows lines which Professor Breslich has long advocated in articles in the *School Review* and elsewhere and lines which the National Committee on Mathematical Requirements accepted in its report of 1923. It is full of exercises suited to the interests and intellectual abilities of adolescent children. It is a true fusion of arithmetic, geometry and algebra. It has a background of careful sifting through practical use and criticism from a large number of teachers and students of the junior high school.

This book is to be followed by another, designed for use in the later years of the junior high school. The second book is completed and is equally based on trial and criticism. This and the second volume of the

series are presented by the institution which Professor Breslich represents, with the full confidence of his associates who have watched the development of his work and shared, to some extent, in its criticism.

August 7, 1924 CHARLES H. JUDD.

JUNIOR MATHEMATICS

CHAPTER I

WHAT IS MEANT BY A LINE-SEGMENT

How to Measure Segments with a Ruler

1. The importance of measurement. Mathematics is one of the earliest sciences. People have found it necessary to measure. Even the savages had to learn how to count and to measure their food supplies and other necessities. This was the beginning of mathematics. At first people measured in a very crude way by means of fingers or pebbles, but as civilization progressed, better and more accurate methods of measurement were needed and therefore developed. Today, everybody uses some kind of measurement in his daily tasks. Without the use of mathematics, we could not construct our modern machinery, our great buildings, railroads, bridges, or ships. We could not even carry on our daily business. Since modern civilization owes so much to mathematics and depends on it so largely, every pupil should study this subject.

Measurement plays a large part in mathematics because it is something everybody needs to know about. The two boys in Fig. 1 are using measurement in laying out a tennis court.

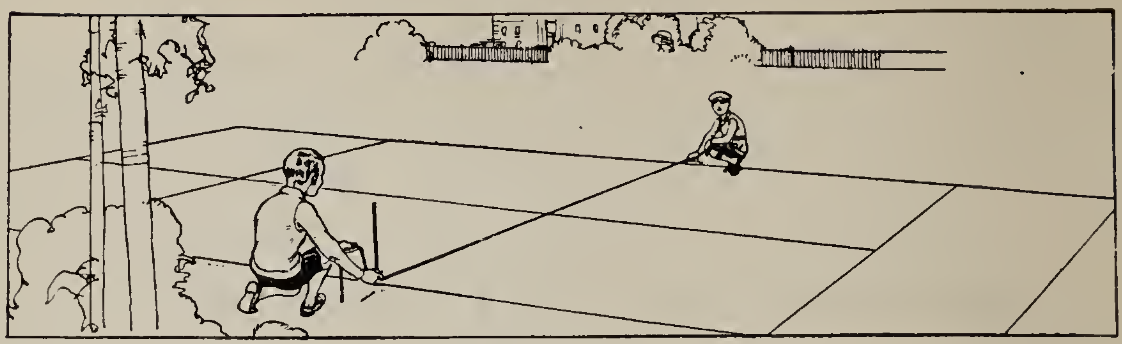


FIG. 1

A boy who plans to make a radio set, a footstool, a kite, or a canoe must determine by measurement the amount of material he needs. A girl who is going to make an apron, a doily, a tablecloth, or a dress, determines by measurement the required amount of material before buying it. She knows from experience that failure to measure may cause waste of material and money if she should purchase more than she needs.

In making a cake it is safer to use a measuring cup and scale than to guess at the amount of the ingredients. When we are sick, the doctor measures our temperature with a thermometer. We use clocks to measure the time, a gas meter to tell the amount of gas we use, a speedometer to show how fast our car travels, and a steam gage to determine the pressure in our heating plant.

The accuracy required in measurement depends largely on the use to be made of the result. If a boy wants to measure the length of the block in which he lives, he may determine it by simply stepping it off; but the surveyor needs to measure the block with great care using as an instrument of measurement—a well made steel tape. A carpenter can measure sufficiently well with a yard stick or tapeline, but the

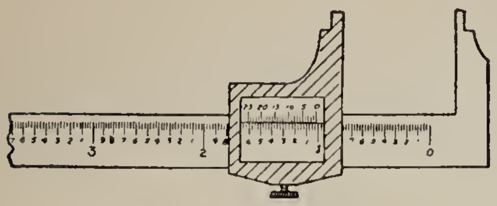


Fig. 2

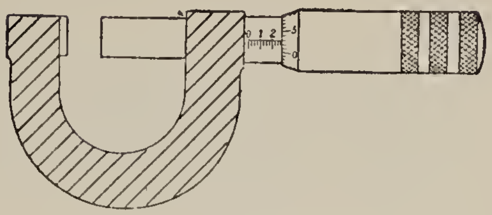


Fig. 3

machinist finds it necessary to use a vernier caliper (Fig. 2) and a micrometer (Fig. 3) to measure a required length.

Exercises

1. Make a list of the measuring instruments used in your household, and tell how each is used.

2. If you are studying science, make a list of the measuring instruments used in that course and tell what they are used for.

3. Name some measuring instruments that are used in stores, offices, and factories.

2. The meaning of a straight line. We shall begin measuring by finding lengths of straight-line distances, because lengths laid off on straight lines are very simple to measure. A very good idea of what is ordinarily called a straight line may be obtained from the following:

Fold a sheet of paper and crease it by moving a finger along the fold. The crease of the paper represents a *straight line.* Boundary lines are frequently straight lines. Thus, the edge of a good ruler, the edges of a sheet of notebook paper, and the boundaries of a window pane are examples of straight lines. When we study lines in *geometry* we do not consider width, thickness, color, or weight. Geometric lines have only length, and we are interested mainly in measuring lengths.

EXERCISES

1. Point out illustrations of straight lines in the classroom.

2. State some examples of straight lines found outside of the classroom.

3. Can you mention some examples of straight lines found in nature?

3. How to make drawings representing straight lines. Various instruments may be used to make drawings representing straight lines. Fig. 4 is a picture of a *ruler*. The edge AB is a straight line.

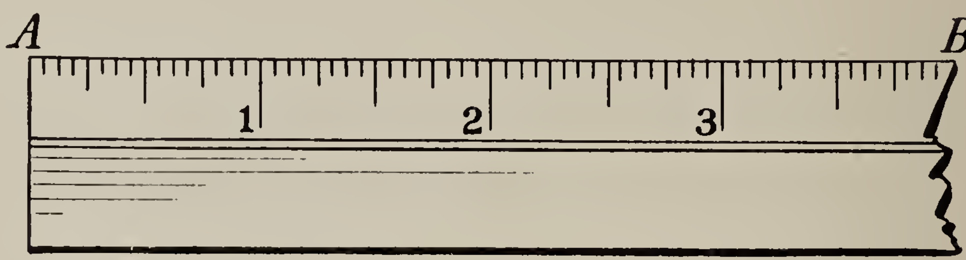


FIG. 4

If the marks or graduations are left off on a ruler, it is called a *straight edge* (Fig. 5).

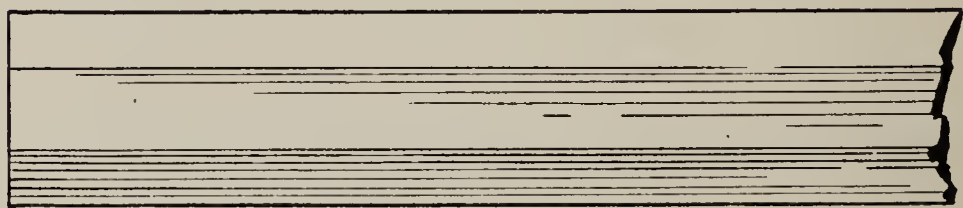


FIG. 5

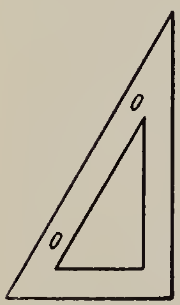


FIG. 6

Other instruments used to draw straight lines are the *triangle* (Fig. 6) and the *T-Square* (Fig. 7).

We shall now learn how to use the ruler in drawing straight lines.

FIG. 7

Exercises

1. Place a ruler on a sheet of paper (Fig. 8) and move the point of a sharp pencil along the edge making a line on the paper. The drawing obtained is said to *represent* a straight line. To be brief, we shall call it a straight line, but strictly speaking, it is not a geometrical line because it has width.

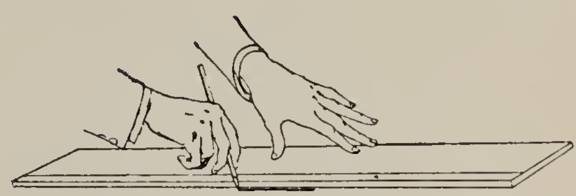


Fig. 8

2. The drawings (Fig. 9) are formed by straight lines. Study

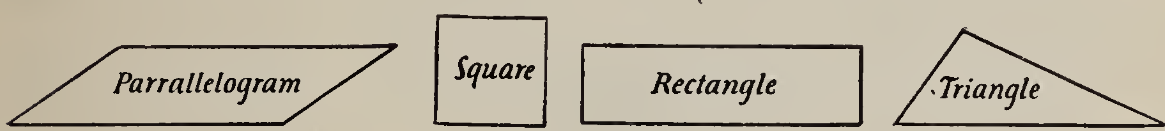


Fig. 9

the shapes and remember the names. Then, without looking at the figures, draw others like them on a sheet of notebook paper.

Find other figures of such shapes in the classroom.

3. Make drawings like those shown in Fig. 10.

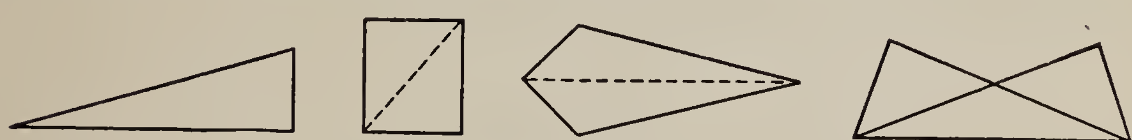


Fig. 10

4. By means of a ruler or straight edge, test whether or not a surface is *plane* (flat).

Suggestion: Place the straight edge of the ruler on the surface in a number of positions (Fig. 11). If for every possible position the edge lies completely on the surface, the latter is said to be *plane.* Apply this test to your desk; to a table top. A carpenter when making a plane surface uses this method of testing.

4. Points. Small dots made on paper with a sharp pencil, or chalk dots made on the blackboard, are ordinarily used to *represent* points. Two geometric lines cut each other in a point, such as point A (Fig. 12). Since geometric lines do not have width, it follows that *geometric points* do not have length, breadth, or thickness. They only indicate position. Points are usually denoted or named by *capital* letters.

Fig. 11

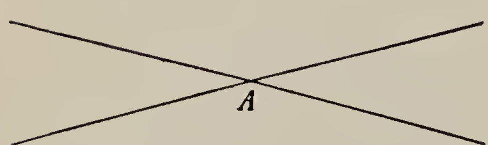


Fig. 12

Exercises

1. Draw two *intersecting* (crossing) straight lines. From the figure tell how many times two straight lines can intersect.

This exercise illustrates the fact that *two straight lines can have only one point in common.* The point in which two straight lines intersect is their *point of intersection.*

2. Draw four straight lines passing through a fixed point A. How many straight lines may be drawn through A?

This exercise illustrates the fact that *through a given point any number of straight lines may be drawn.*

FIG. 13

3. We know from geography that lines drawn on a map from right to left are *east-west* lines. *East* is to the *right*, *west* is to the *left*. Make a drawing like Fig. 13, and through each of the points A, B, C, D, E, draw the east-west line.

4. In the kind of drawing or diagram which we call a map, *upward* means *north*, or toward the top of the page, *downward* means *south*, or toward the bottom of the page. Make a drawing like Fig. 14, and through each of the points A, B, C, D, E, draw the north-south line.

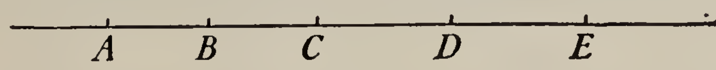


FIG. 14

5. To draw straight lines it is necessary to have a good ruler. Test the straightness of your ruler as follows:

On the blackboard or on a sheet of paper mark two points, A and B.

Placing the ruler upon the paper so that the edge passes through A and B, draw a line through A and B.

Then place the ruler on the opposite side of the line AB, making the edge again pass through A and B, and draw a line through A and B.

If the edge of the ruler is straight and if the drawing is well made, the second line should fall exactly on the first. The two lines are then said to *coincide* (fall together).

This exercise illustrates the fact that *through two given points only one straight line can be drawn.*

5. **Line-segment.** In the preceding paragraphs the word "line" has been used without considering length. A geometric line is unlimited in length.

FIG. 15

A *limited portion* of a line, *i.e.*, one which is *bounded* by two points like A and B (Fig. 15), is a *line-segment*.

Note that the difference between line-segment AB and line AB is that the first is bounded by two points, while the second extends indefinitely.

6. How to measure length. Lengths may be measured with various instruments. To measure the length of a line-segment as AB (Fig. 16) with a ruler,

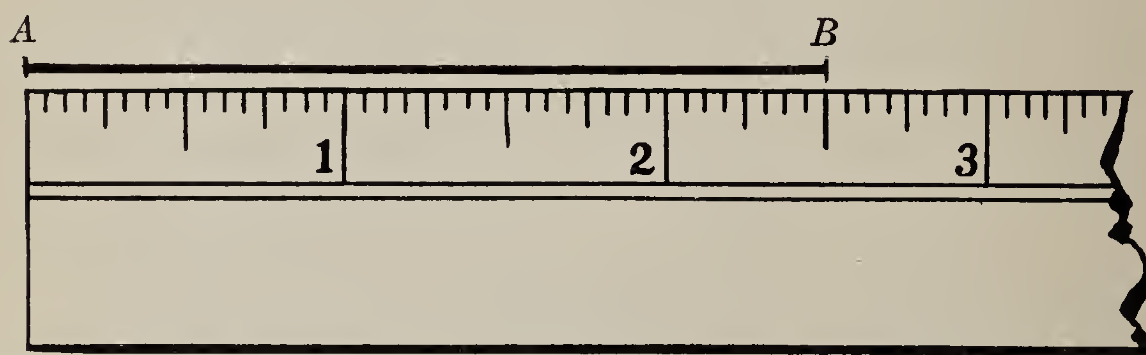


FIG. 16

place the edge, which is marked off into equal parts, along AB with the zero mark directly under A.

Then read off the mark on the ruler which is directly under B. If the ruler is graduated in inches the reading gives the number of inches contained in AB. This number is the *length* of AB, and the inch is the *unit-segment.* Thus, to *measure* a line-segment is to determine how many times it contains another segment, called the *unit-segment.*

EXERCISES

1. Look again at Fig. 16 and tell the length of AB.

2. Draw a line-segment and find the length by measuring with a ruler as you were shown in §6.

In this exercise some of you have discovered that when we are measuring segments, we cannot always find the *exact* length, because the end point of the segment does not in every case fall

exactly over a mark of the ruler. The length then has to be *estimated.* Thus, in Fig. 17 the length of AB is greater than $2\frac{7}{16}$ and less than $2\frac{1}{2}$. It seems to be nearest to $2\frac{7}{16}$. The length is said to be $2\frac{7}{16}$, *approximately,* or $2\frac{7}{16}$ *to the nearest sixteenth of an inch.*

Briefly, $AB = 2\frac{7}{16}$ approximately.

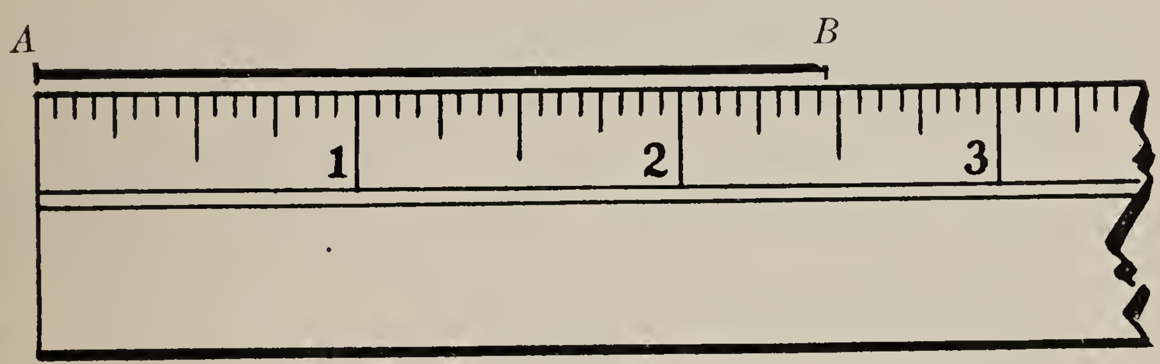


FIG. 17

3. State how one measures a line-segment with a ruler.

4. Draw a line-segment and find the length to the nearest sixteenth of an inch. Write the result in the form used in the final statement in Exercise 2.

5. Measure each of the segments AB, BC, CA (Fig. 18). Write the results arranged as follows:

$AB =$

$BC =$

$CA =$

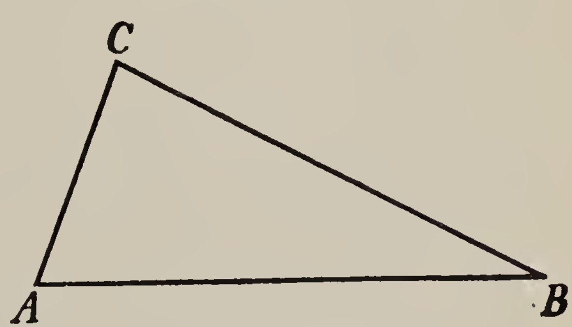


FIG. 18

Add the lengths of the three segments and divide the sum by 3.

6. Make a drawing like Fig. 19. Measure each of the three segments. Find the sum and divide it by 3. Arrange the work as in Exercise 5.

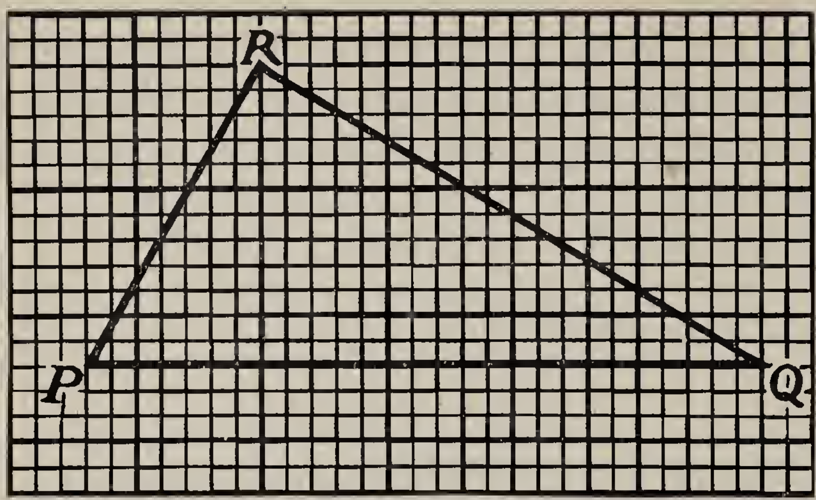


Fig. 19

7. Draw five segments and name them AB, CD, EF, GH, IK. Measure each to the nearest sixteenth of an inch. Find the sum and divide the sum by 5.

8. Make a drawing like the rectangle in Fig. 9 and measure each side to the nearest sixteenth of an inch.

9. Draw a square (Fig. 9) and measure each side to the nearest sixteenth of an inch.

7. Arithmetic average. In a problem in measuring, the results of all pupils in a class usually do not agree exactly. Some pupils are more accurate than others. If we compare classes, we find that some classes are more accurate than others. If we add the results when all pupils in a class have measured a line-segment, and then divide the sum by the number of pupils in a class, we obtain the *average length*, or the arithmetic average. Likewise, the arithmetic average of *several* segments is found by adding the lengths of all the segments and dividing the sum by the number of segments. The average of the results obtained by all the pupils in a problem in measurement is used to tell how accurately the class can measure. We may also find the separate averages for the boys and the girls and, by comparing each of these averages with the correct measure, we can compare the work of the boys with that of the girls.

The following example shows how to find the average of several numbers.

Find the average of $6\frac{2}{3}$, $4\frac{1}{2}$, $5\frac{3}{4}$.

Solution: Changing all fractions to the same denominator, we have

$$\begin{aligned} 6\tfrac{2}{3}+4\tfrac{1}{2}+5\tfrac{3}{4} &= 6\tfrac{8}{12}+4\tfrac{6}{12}+5\tfrac{9}{12} \\ &= 16\tfrac{11}{12} \end{aligned}$$

To find the average, divide $16\frac{11}{12}$ by 3.

$$\begin{aligned} \text{Average} &= \tfrac{1}{3}\times 16\tfrac{11}{12} = \tfrac{1}{3}\times\tfrac{203}{12} \\ &= \tfrac{203}{36} \\ &= 5\tfrac{23}{36} \end{aligned}$$

$$\text{Average} = 5\tfrac{23}{36}$$

Computation:

$$\begin{array}{r} 16 \\ \underline{12} \\ 32 \\ \underline{16} \\ 192 \\ \underline{11} \\ 203 \end{array}$$

$$\begin{array}{r} 5 \\ 36\overline{)203} \\ \underline{180} \\ 23 \end{array}$$

To add the fractions $\frac{2}{3}$, $\frac{1}{2}$, and $\frac{3}{4}$ in the example above, the least common multiple of the denominators has to be found. To find the least common multiple of 3, 2, and 4, we write the largest of these numbers as 4, and then multiply it successively by 2, 3, etc., until a multiple is found containing 3, *i.e.*, 4×3, or 12. Since this contains the remaining number, 2, it is the least common multiple of 4, 3 and 2.

Similarly, to find the least common denominator of $\frac{3}{16}$, $\frac{17}{48}$, and $\frac{8}{9}$ we write down the largest denominator,

48. Since 48 contains 16, we disregard the 16. Multiplying 48 by 2, 3, etc., we see that 3×48, or 144, contains 9. Thus 144 is the least common denominator.

Exercises

For each of the columns below find the average. Arrange the work as shown in the preceding example.

1. $8\frac{1}{2}$	2. $32\frac{7}{8}$	3. $16\frac{2}{3}$	4. $8\frac{1}{2}$	5. $22\frac{3}{16}$
$10\frac{1}{4}$	$20\frac{3}{4}$	$15\frac{1}{3}$	$15\frac{3}{4}$	$17\frac{29}{48}$
$15\frac{2}{8}$	$12\frac{21}{24}$	$19\frac{5}{6}$	$22\frac{3}{8}$	$10\frac{8}{9}$
	$4\frac{5}{6}$		$7\frac{2}{3}$	

6. In measuring a line-segment six pupils of a class found the following results:

$4\frac{1}{2}$, $4\frac{7}{16}$, $4\frac{9}{16}$, $4\frac{3}{8}$, $4\frac{7}{16}$, $4\frac{1}{2}$. Find the average.

7. Draw five segments, each being longer than the preceding. Name them AB, CD, EF, GH, and IK. Measure each segment to the nearest sixteenth of an inch, and subtract their lengths in the following order: the first from the second, the second from the third, the third from the fourth, and the fourth from the fifth. State your results in the following form.

$$CD-AB= \quad , \qquad IK-GH= \quad ,$$
$$EF-CD= \quad , \qquad EF-AB= \quad ,$$
$$GH-EF= \quad , \qquad GH-CD= \quad .$$

Exercises 8 to 28 give further practice in such additions and subtractions as occur in the preceding exercises.

8. In the following subtract the lower number from the upper. Check each by adding the lower to the difference.

$$\begin{array}{r} 12\frac{1}{4} \\ \underline{7\frac{2}{3}} \end{array}$$

Solution:

$$\begin{array}{rcl} 12\frac{1}{4} = 12\frac{3}{12} &=& 11\frac{15}{12}. \\ 7\frac{2}{3} = 7\frac{8}{12} &=& 7\frac{8}{12}. \\ \hline \text{difference} &=& 4\frac{7}{12}. \end{array}$$

Check:

$$7\frac{8}{12}+4\frac{7}{12}=11\frac{15}{12}$$
$$=12\frac{1}{4}$$

9. In each of the following subtract the lower from the upper number:

$12\frac{3}{16}$	$18\frac{7}{12}$	$50\frac{1}{3}$	$7\frac{1}{2}$	$68\frac{3}{4}$
8	$10\frac{1}{3}$	$14\frac{11}{12}$	$5\frac{3}{4}$	$11\frac{45}{48}$
$9\frac{7}{8}$	14	$8\frac{2}{7}$	$22\frac{1}{3}$	$14\frac{1}{2}$
$5\frac{2}{3}$	$8\frac{2}{3}$	$4\frac{4}{5}$	$9\frac{3}{5}$	$7\frac{1}{3}$

Find the following sums and differences. Arrange all work as shown in Exercise 10.

10. $\frac{1}{2}+\frac{3}{4}-\frac{1}{5}$

Solution: $\frac{1}{2}+\frac{3}{4}-\frac{1}{5}=\frac{10}{20}+\frac{15}{20}-\frac{4}{20}=\frac{21}{20}=1\frac{1}{20}.$

11. $\frac{1}{3}+\frac{1}{4}.$	**16.** $\frac{7}{12}-\frac{1}{5}.$	**21.** $2\frac{1}{2}+2\frac{2}{3}-1\frac{1}{8}.$
12. $\frac{1}{4}-\frac{1}{6}.$	**17.** $\frac{9}{10}-\frac{2}{5}.$	**22.** $4\frac{4}{7}+6\frac{7}{8}-2\frac{1}{2}.$
13. $\frac{5}{6}+\frac{7}{9}.$	**18.** $1\frac{2}{3}+3\frac{1}{2}.$	**23.** $10\frac{1}{2}-8\frac{1}{3}+5\frac{3}{4}.$
14. $\frac{7}{12}-\frac{1}{3}.$	**19.** $2\frac{1}{2}+1\frac{3}{8}.$	**24.** $18\frac{3}{16}-10\frac{4}{5}+\frac{1}{2}.$
15. $\frac{7}{8}-\frac{1}{4}.$	**20.** $8\frac{1}{8}-6\frac{1}{4}.$	**25.** $16+2\frac{4}{7}-3\frac{1}{3}.$

26. An empty soap box weighs $4\frac{1}{4}$ pounds. It is packed with bars of soap weighing $54\frac{1}{8}$ pounds. By adding $4\frac{1}{4}$ and $54\frac{1}{8}$ find the weight of the box after packing.

27. The sum of two numbers is $30\frac{1}{8}$. One of the numbers is $16\frac{3}{4}$. Find the other by subtracting $16\frac{3}{4}$ from $30\frac{1}{8}$.

28. A table top 32 in. wide is to be made by glueing together 4 boards. How wide must be the fourth board if the widths of the others are respectively, $7\frac{7}{8}$ in., $7\frac{3}{4}$ in., and 8 inches?

29. In the drawing (Fig. 20) measure the length and the width of the frame of the picture, each to the nearest sixteenth of an inch. Add the length to the width.

30. In making a drawing of an object architects, designers, surveyors, and draftsmen usually do not draw a full size picture. Each line in the drawing is made a specified part of the actual length.

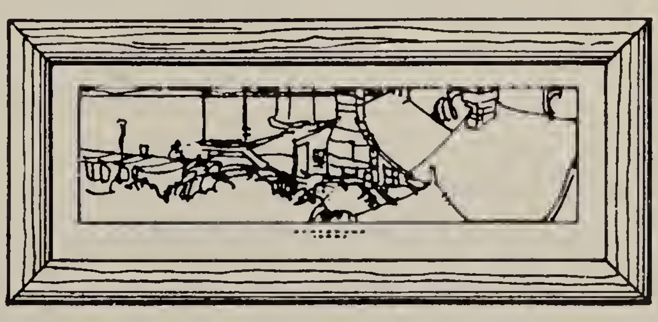


Fig. 20

The object is then said to be *drawn to scale.* The following exercises show how actual length can be found in a scale drawing.

If an inch on the drawing (Fig. 20) represents a length of 8 inches in the frame of the picture, the actual length and width of the frame may be found by multiplying each of the measures in Exercise 29 by 8. Find the actual length and width of the frame.

Fig. 21

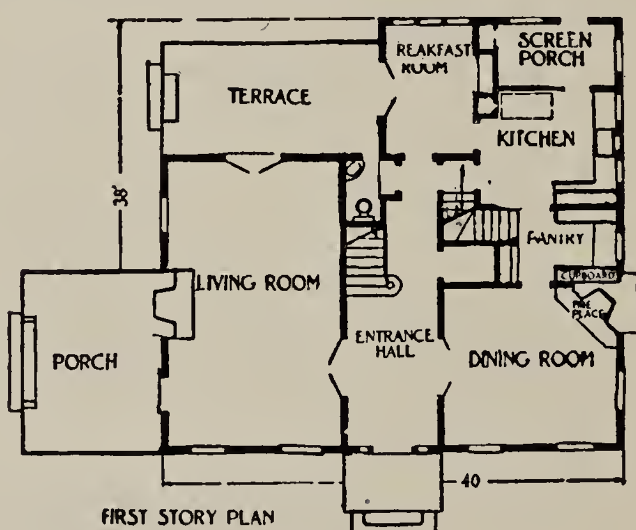


Fig. 22

31. The arrangement of the rooms on the first floor of a house (Fig. 21) is shown in the drawing (Fig. 22). One inch in the drawing represents 24 feet in the house. Find the approximate dimensions of the living room, dining room, kitchen and porch.

32. Fig. 23 is a map of the State of Illinois. The segment AB in the lower left-hand corner represents a distance of 70 miles. It

FIG. 23

is to be used to measure distances between places on the map to the nearest mile, just as we have used the ruler to measure the distance between the end points of a segment.

Fold a sheet of paper, and on the crease mark off the scale given on *AB*, making a graduated straight edge.

Using this as a ruler, find to the nearest mile the distance of each of the following cities from Chicago: Rockford, Clinton, Quincy, Peoria, St. Louis, Cairo, Danville.

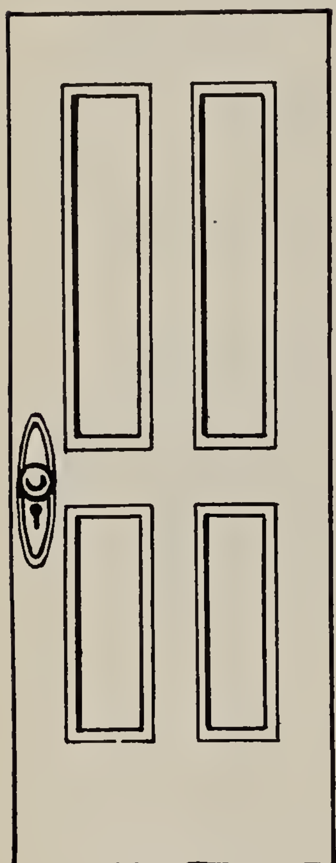


Fig. 24

33. In the drawing (Fig. 24) an inch represents 2 ft., 6 in. Find the dimensions of the door.

Solution: Width $=1 \times 2\frac{1}{2}=2$ ft., 6 inches.

$$\text{Length}=2\tfrac{2}{3}\times 2\tfrac{1}{2}=\frac{8}{3}\times\frac{5}{2}=\frac{\overset{4}{\cancel{8}}\times 5}{3\times\cancel{2}}=\frac{20}{3}=6\tfrac{2}{3}.$$

Length $=6$ ft., 8 inches.

Similarly find the dimensions of the panels.

Exercises 34 and 35 below give practice in such multiplications as occur in Exercise 33.

34. Multiply the following numbers, arranging your work as shown in the first exercise.

$23\frac{1}{6}\times 4$.

Solution: $23\frac{1}{6}\times 4=92\frac{4}{6}=92\frac{2}{3}$.

$\frac{5}{8}\times 7$	$\frac{3}{5}\times 9$	$7\frac{3}{8}\times 16$
$24\frac{2}{3}\times 6$	$\frac{2}{3}\times 15$	$3\times 6\frac{7}{8}$
$6\frac{6}{7}\times 24$	$3\frac{1}{2}\times 5$	$8\frac{2}{3}\times 10$

35. Multiply as indicated the following factors.
$14\frac{2}{3}\times 4\frac{1}{5}$.

$$\textit{Solution: } 14\tfrac{2}{3}\times 4\tfrac{1}{5}=\frac{44}{3}\times\frac{21}{5}=\frac{44\times\overset{7}{\cancel{21}}}{\underset{1}{\cancel{3}}\times 5}=\frac{308}{5}=61\tfrac{3}{5}.$$

The work of writing the solution may be lessened by omitting the step $\frac{44}{3}\times\frac{21}{5}$, but not the step $\frac{44\times\overset{7}{\cancel{21}}}{\cancel{3}\times5}$.

$26\frac{3}{8}\times12\frac{1}{3}$	$48\frac{3}{4}\times2\frac{1}{2}$	$1\frac{3}{8}\times12\frac{1}{2}$
$45\times33\frac{1}{3}$	$3\frac{1}{4}\times2\frac{4}{7}$	$16\frac{1}{7}\times5\frac{1}{4}$
$14\frac{3}{5}\times11\frac{2}{3}$	$21\frac{2}{3}\times3\frac{4}{5}$	$14\frac{1}{3}\times7\frac{2}{5}$
$3\frac{1}{9}\times12\frac{1}{2}$	$16\frac{1}{4}\times12\frac{1}{2}$	$9\frac{2}{5}\times5\frac{5}{6}$
$7\frac{5}{6}\times8\frac{1}{8}$	$8\frac{2}{7}\times11\frac{3}{5}$	$18\frac{3}{4}\times17\frac{1}{3}$
$9\frac{3}{7}\times13\frac{3}{4}$	$6\frac{4}{9}\times9\frac{1}{8}$	$24\frac{1}{2}\times3\frac{1}{8}$

Measuring Line-segments with Ruler and Compass

8. The compass. So far segments have been measured with a ruler alone. Line-segments may be measured with a compass (Fig. 25) as follows:

Open the compass and place the sharp points at A and B, the end points of the segment AB. Then place

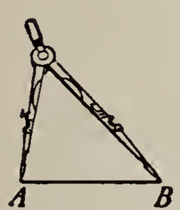


Fig. 25

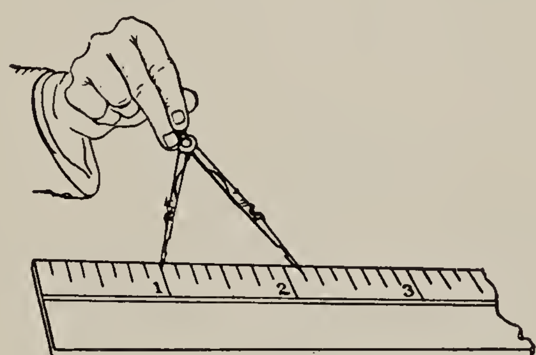


Fig. 26

the points on the marks of a ruler (Fig. 26) and count to the nearest sixteenth of an inch the number of inches between them. This is the *length* of AB. Why?

By this method of measuring with the compass one is able to measure more accurately than with the ruler alone. For, when we measure with the ruler the eye

has to pass from the end points of the segment to the marks on the ruler, which makes it difficult to get the best reading. The error is reduced by carrying with the compass the distance between the end points from segment to ruler. Hence, when *exact* work is required the compass should be used. *It is important to keep the pencil point of the compass sharpened.*

Exercises

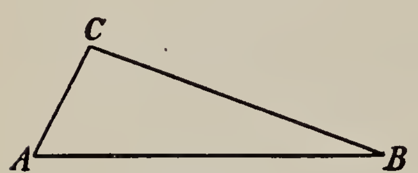


Fig. 27

1. Using the compass and ruler as shown above measure AB, BC, and CA (Fig. 27) to the nearest sixteenth of an inch.

2. The drawings (Figs. 28 and 29) represent a match safe and the designs (working drawings) for making the safe. A length of an

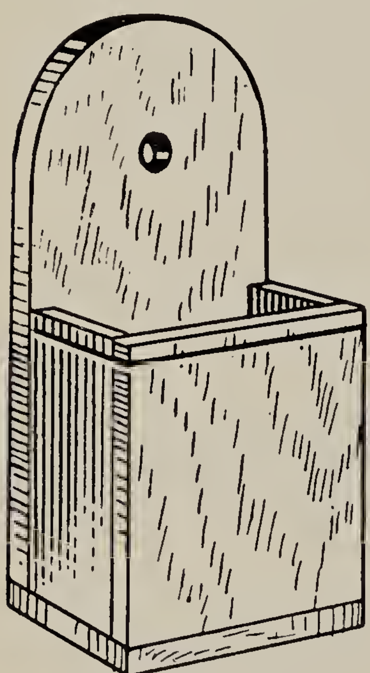


Fig. 28

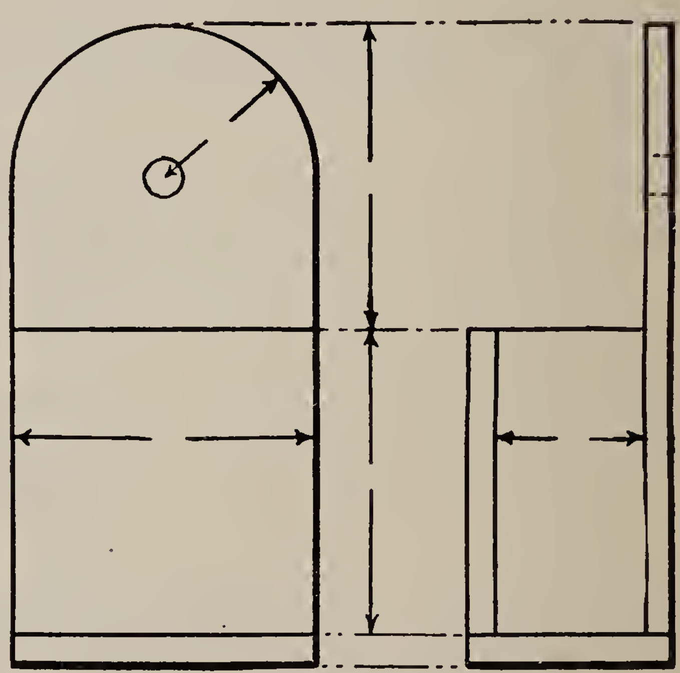


Fig. 29

inch in the design represents 2 inches in the corresponding length of the safe. Find to the nearest sixteenth of an inch the following dimensions of the safe: width, height, depth, and thickness of boards.

9. How standard units of length have been made. Most of the civilized nations have derived a unit of length from the length of the human foot. The result has been that standard units of length are not the same in different countries.

Early units of measurements were indefinite. Thus the *foot-breath* and *cubit* (distance from the elbow to the extremity of the middle finger) mentioned in the Bible vary for different persons, and exact measurement with these units is impossible. Hence it became necessary for people to adopt a definite unit.

In a book on surveying we find the following account of how people in the sixteenth century tried to obtain a standard unit. "Stand at the door of a church on Sunday, and bid sixteen men to stop, tall ones and small ones, as they happen to pass out when service is finished; then make them put their left feet one behind the other and the length thus obtained shall be a right and lawful rod to measure and survey land with, and the sixteenth part of it shall be a right and lawful foot."[1]

Henry I King of England (1100–1135) A.D. is said to have used as a unit the distance from the point of his nose to the end of his thumb, approximately a yard's length. Other units were made in 1490 by Henry VIII and in 1588 by Queen Elizabeth. The present English standard was adopted in 1855.

In 1856 the English Government sent to this country two copies of the new English standard, one made of bronze, the other of iron, which were used as standards

[1] Lessons in Community and National Life. Series B, Chapter VI. By Charles H. Judd and Leon C. Marshall.

until 1875. The table below gives the units of linear measure in that system.

TABLE OF LINEAR MEASURE IN THE ENGLISH SYSTEM

12 inches (in.)	=1 foot (ft.).
3 feet (ft.)	=1 yard (yd.).
$5\frac{1}{2}$ yards = $16\frac{1}{2}$ feet	=1 rod (rd.).
320 rods = 5280 feet	=1 mile (mi.).
6 feet	=1 fathom.
1.151 miles	=1 knot.

During the French Revolution the National Assembly (1790) appointed a committee of the Academy of Sciences to study the matter of finding a suitable system of weights and measures. This commission selected as the standard unit of length the one-ten-millionth part of the distance from the North Pole to the equator, measured along the meridian through Paris. This standard unit is called *meter*. The commission determined the length of a meter to be about 39.37 inches, the work requiring seven years for its completion. This distance was marked off by expert instrument makers on a bar of platinum. On June 22, 1799, the standard unit was presented to the Council of Five Hundred and deposited in the archives at Paris. It is the only unit of length which is the result of scientific investigation. The meter (m.) is divided into 10 equal parts called *decimeters* (dm.). Each decimeter is divided into 10 equal parts called *centimeters* (cm.). Each centimeter is divided into 10 equal parts called *millimeters* (mm.). A thousand meters make a *kilometer* (km.). The following table expresses these lengths in terms of inches, feet, yards, and miles.

TABLE OF LINEAR MEASURE IN THE METRIC SYSTEM

	Inches	*Feet*	*Yards*	*Miles*
Millimeter...............	0.03937	0.003	0.001	
Centimeter..............	0.3937	0.033	0.011	
Decimeter...............	3.937	0.328	0.109	
Meter....................	39.37	3.281	1.093	
Kilometer...............	39370.	3280.833	1093.611	0.621

We shall speak of this system of measures as the *metric system.* It is now generally used throughout the world in scientific investigations. In view of the trade between nations it is desirable that they employ a uniform system of measures in commerce. In 1866 a law was passed making the metric system legal in the United States. The system is easier to understand and to use than our common system. Fig. 30 represents

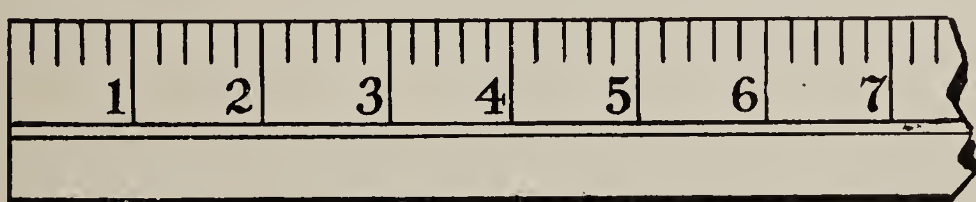


FIG. 30

a part of a ruler graduated according to the metric system.

The following editorial, taken from one of our leading newspapers, describes the merits and importance of the metric system, in urging its adoption:

ADOPT THE METRIC SYSTEM

One of the bills which ought to be passed by congress is the Britten-Ladd bill, which provides for a gradual adoption of the metric system in the United States. The measure has been well considered and is conserviatve in its provisions. It provides for a

transition period of ten years in merchandising, leaving manufacturers to adopt the new measures as they please. It is certain that if given a footing metric standardization will approve itself and the existing measures will be abandoned without serious disarrangements or expenditures within a comparatively short time.

The metric system is now used by most modern advanced peoples, ourselves and the British being the important exceptions. It gives easier and in practice more accurate measure than the Anglo-American traditional units of measure, as many an American soldier discovered during the war. But even if it were not more scientific, its adoption would be advisable for us because it is the standard in use in the markets which Americans hope to enter throughout the world. Our present system is a substantial handicap for our foreign trade, in South America for example. Differences in measure of length, capacity, and weight are annoying to our customers and a deterrent to the purchase of American goods.

In proportion as the development of foreign trade is essential to our prosperity the need for accepting the international standard becomes urgent. This is important in peace. In case of war it is even more important. In the late conflict our system of measures was a serious obstacle to prompt and effective coöperation and exchange of resources with our allies. In consequence, men like Gen. Pershing urge adoption of the metric standards. In fact, soldiers, scientists, educators, manufacturers, and commercial men of the highest standing are the emphatic advocates of international standardization upon the metric basis. But the reform is of importance to all of us in proportion as we are all affected, directly or indirectly, by the expansion and efficiency of our trade, both domestic and foreign.

The reform has been on the way too long. Our foreign commerce cannot afford any handicap of which it can rid itself. It will take time to put the system in operation and make the necessary adjustments and it is therefore bad policy to postpone action longer.

Exercises

1. In our reading we find mention of such units as: cubit, pace, fathom, hand, ell. Find out what these units mean and how they originated.

2. Using the metric scale on your ruler, measure the length of the page of the textbook to the nearest tenth of a centimeter.

3. Change 268 millimeters to centimeters; 3764 mm. to meters; 8 m. to decimeters; 15 m. to decimeters; 2486 mm. to centimeters.

4. The distance between two cities in France is 132 kilometers. How many miles are they apart?

5. The height of an airplane is 80 meters. Express this in feet.

6. The speed of an airplane is 68 kilometers. Express the speed in miles.

7. By means of the compass determine to the nearest sixteenth of an inch the number of inches contained in 3 centimeters; 5 centimeters; 6 centimeters; 8 centimeters. Arrange your results in the form of a table.

8. The distance between two cities in France is 153 kilometers. Express this in miles.

Measuring Line-segments with Squared Paper

10. Squared paper. Fig. 31 represents a part of a sheet of squared paper. It is ruled with east-west and north-south lines. They divide the paper into large and small squares. By measuring with the ruler we find the sides of the large squares to be one centimeter long, the sides of the small squares are .2 of a centimeter.

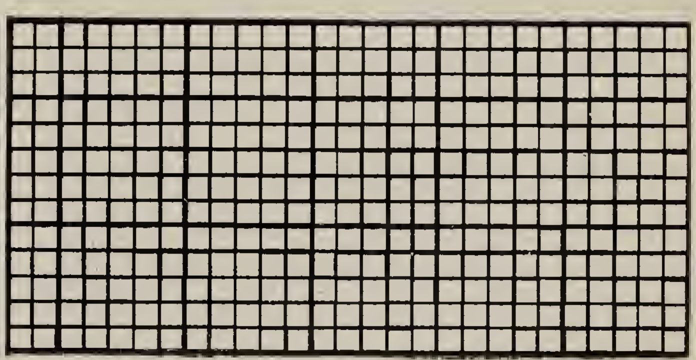


Fig. 31

Thus the lines on the paper are divided according to the *metric system*.

11. How to measure with squared paper. Squared paper may be used to measure segments. It is convenient to select as a unit a segment 2 centimeters long, as AB (Fig. 32).

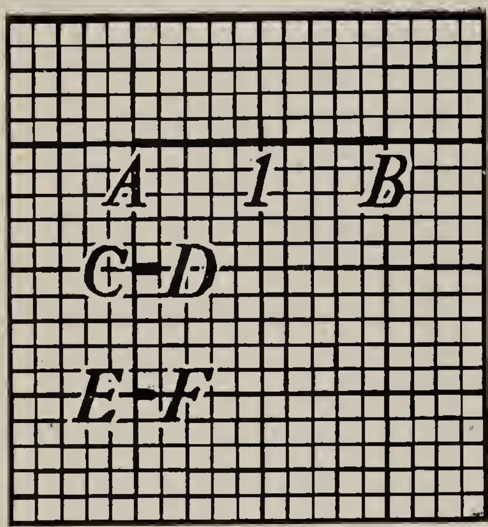


FIG. 32

Then the length of CD is one-tenth of that of AB, or .1.

With a little practice we shall be able to estimate the length of a segment which is shorter than CD. Thus EF is less than CD, but greater than one-half of CD. To determine the length of EF with a fair degree of accuracy *imagine* CD divided into 10 equal parts. Each of these parts is one-tenth of CD, and therefore equal to one-hundredth of AB, or .01. Similarly, one-half of CD is equal to .5 of CD, or .05.

By examining the segment EF we find that it is greater than .05, but less than .1. It seems to be about .08.

Thus, we have seen that if $AB = 1$, then $CD = .1$, and $EF = .08$, approximately. We shall now learn to measure segments, using a unit 2 cm. long.

EXERCISES

1. Measure AB (Fig. 33).

Directions: Open the compass placing the sharp points, at A and B, respectively. Place the metal point of the compass on one of the corners of a large square, such as C.

With the pencil point make a mark on one of the heavy lines passing through C. This locates the point D.

Then CD is of the same length as AB, and the length of AB may now be found by measuring CD.

Show that CE is equal to 1.

Show that EF is equal to 1.

Hence CF is equal to 2.

Show that FG is equal to .2.

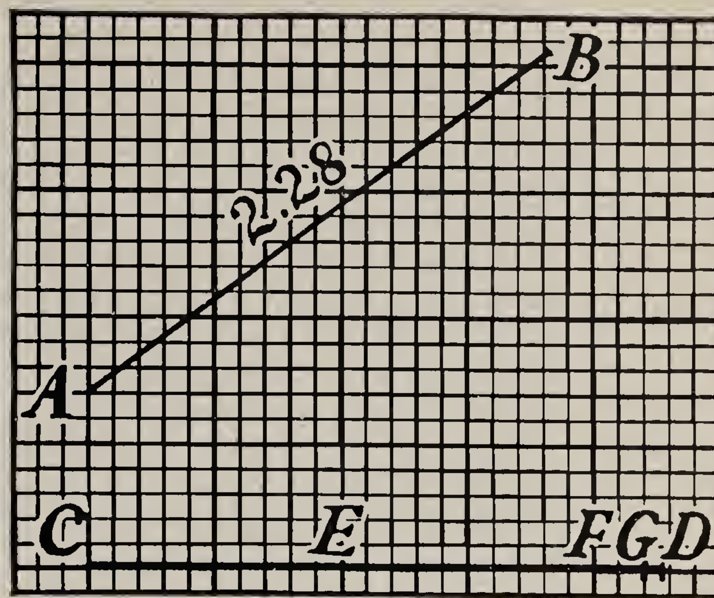


Fig. 33

Show by estimating that GD is .08, approximately.

Show that CD is approximately 2.28.

Hence AB is *approximately* 2.28.

After measuring a segment write the length on the segment, as shown in Fig. 33.

Notice that in this result the first and second figures in the number 2.28 are *exact*, but that the last figure is *uncertain*. We say that AB has been *measured to three figures* or to two decimal places.

2. Show that the length of AB (Fig. 34) is 0.84, when measured to three figures. This result is stated in the form

$AB = 0.84$ approximately.

3. Draw a segment and measure it to three figures. State the result in the form shown in Exercise 2.

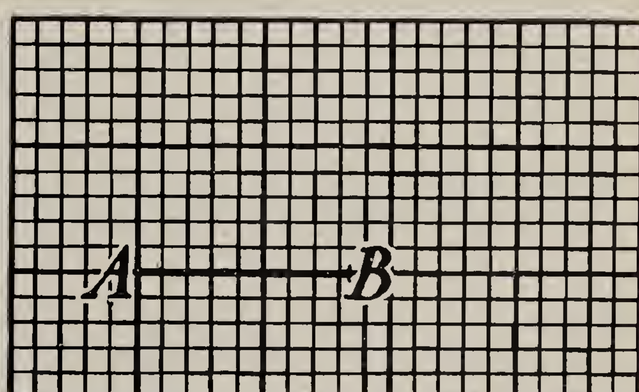


Fig. 34

4. Measure to three figures the segment AB (Fig. 35). Let one pupil write the results of several others on the blackboard. Let each pupil find the average of these results. Which results differ least from the average?

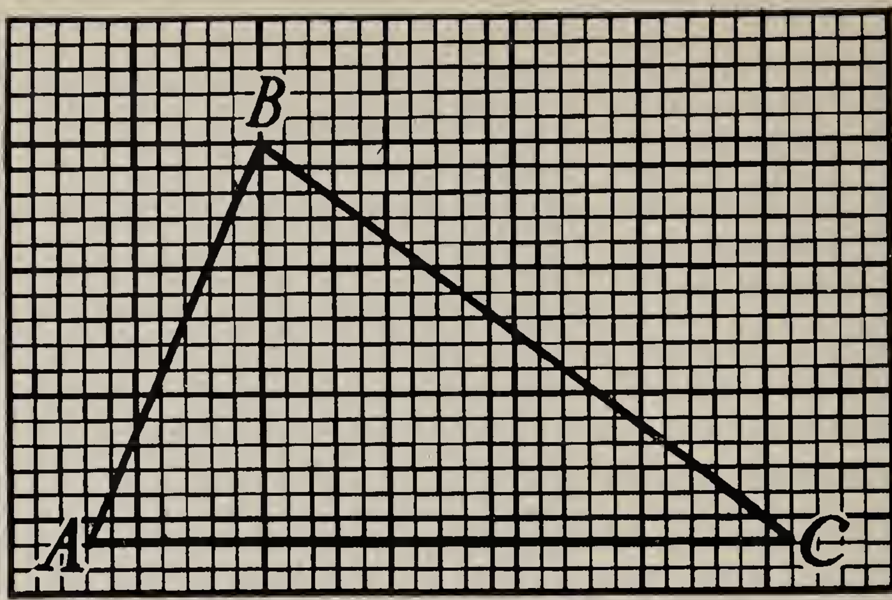


Fig. 35

Measure AC and BC (Fig. 35) each to tnree figures.

5. Measure to three figures each of the segments AB, BC, CD, and DA (Fig. 36). State the results as in Exercise 2.

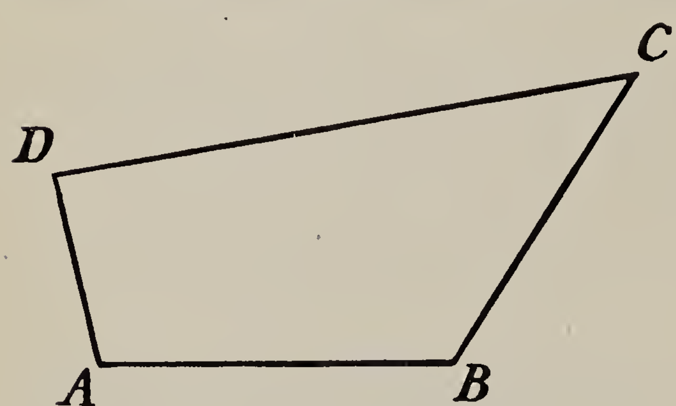


Fig. 36

6. Draw segments AC and DB (Fig. 36), and measure each to three figures.

7. Measure to three figures the distances from A to B (Fig. 37), from B to C, from B to D, from A to C. State the results as in Exercise 2.

12. Symbols of equality and inequality. In §11 we used the statements *is equal to*, *is greater than*, and *is less than*. In mathematics it is convenient and customary to use symbols to denote briefly such verbal statements. Thus,

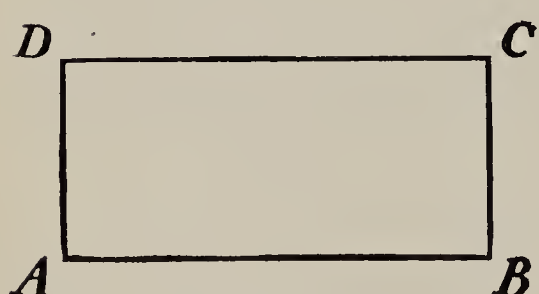


Fig. 37

5 is equal to 4+1 is written $5=4+1$;
6 is less than 8 is written $6<8$;
7 is greater than 5 is written $7>5$.

Thus, $=$ means "is equal to"; $>$ means "is greater than"; $<$ means "is less than."

The pupil must make himself familiar with the meaning of these symbols.

13. Equation. A statement expressing the equality of two numbers, as $5=4+1$, $a=3.24$, is called an **equation.**

Exercises

Write each of the following statements in symbols: five is greater than three; eight is equal to the sum of six and two; seven is less than ten; AB is less than MN; x is greater than y; the difference between a and b is less than the sum of a and b.

14. Notation for line-segments. We have denoted line-segments by marking the end points with *capital* letters. Sometimes one *small* letter is used to denote a line-segment. This is written on the segment near the mid-point, as a (Fig. 38).

Fig. 38

Capital letters usually represent *points, small* letters denote *numbers*. In Fig. 38 the number a is an *unknown* number. It denotes the *length* of the segment, and may be found by measuring the segment.

Exercises

1. Measure AB (Fig. 38), and thus find the number denoted by the small letter a.

2. By measuring find the numbers denoted by a, b, and c (Fig. 39).

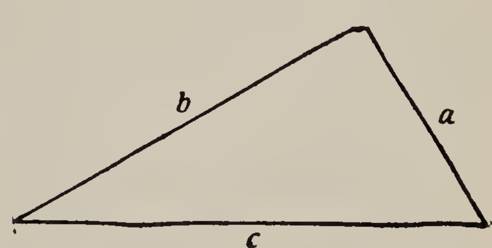


Fig. 39

15. Literal number. There are various ways of denoting numbers by symbols. For example, the number five may be denoted as follows: 5, V, ∴, or 𝍸. A number denoted by a letter is a **literal number.**

COMPARING LENGTHS OF SEGMENTS BY MEANS OF RATIOS

16. How to find the ratio of line-segments. Let AB and CD be two line-segments whose lengths are respectively, 2.38 and 3.15, and let it be required to compare the length of AB with that of CD.

We have $AB = 2.38$ in.
and $CD = 3.15$ in.

Dividing the length of AB by that of CD, we have the quotient

$$\frac{AB}{CD} = \frac{2.38}{3.15} = 0.75$$

$$\therefore \frac{HB}{CD} = 0.75$$

Computation:

$$\begin{array}{r} 0.755 \\ 315\overline{)2380} \\ 2205 \\ \hline 1750 \\ 1575 \\ \hline 1750 \\ 1575 \\ \hline \end{array}$$

This result is not *exactly* equal to $\frac{2.38}{3.15}$, since we have dropped all figures after the second decimal place. However, it is true *approximately to the third figure*, or to *two decimal places*.

The quotient .75 is the *ratio* of AB to CD. This ratio compares segment AB with segment CD. It indicates that AB is about .75 of CD or $\frac{3}{4}$ as long as CD.

In general, the **ratio** of two segments is the quotient found by dividing the measure of one by that of the other, provided a common unit is used in measuring.

Thus, if one segment is 2 in. long and another 3 in., the *ratio* of the segments is $\frac{2}{3}$, and the length of the first segment is $\frac{2}{3}$ of the length of the second.

Exercises

1. Draw two segments, AB and CD. Measure the segments to three figures. Find the ratio by dividing one measure by the other. Arrange your work as shown in §16. State your results approximately to two decimal places.

2. If $AB=3.16$ and $CD=1.24$ find the ratio to three figures, arranging your work as in the example of §16.

3. In triangle ABC (Fig. 40) measure each of the segments AB, BC, and CA to three figures and find, to the third figure, the ratios

$$\frac{a}{b}, \frac{b}{c}, \frac{c}{a}, \frac{b}{a}, \frac{c}{b}, \frac{a}{c}.$$

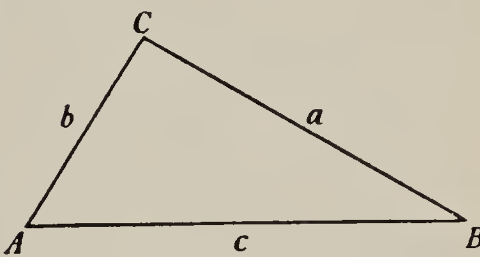


Fig. 40

4. Divide to three figures as indicated:

264 by 681	1.03 by 6.17	23.8 by 24.1
301 by 126	0.15 by 8.34	1.40 by 915
615 by 305	9.00 by 0.02	0.12 by 218

17. How to find the ratio of two numbers. We have seen that lengths may be compared by means of ratios. Two *numbers* may be *compared* with each other by dividing. Thus, to compare 8 with 12 we divide 8 by 12. The comparison is then expressed by the ratio $\frac{8}{12}$ or $\frac{2}{3}$, meaning that 8 is $\frac{2}{3}$ of 12.

In the following table compare each number in the first line with the corresponding number in the second by dividing the number in the first by the corresponding number in the second.

6 in.	2 in.	8 lb.	4 lb.	18 hr.	\$16
2 in.	6 in.	4 lb.	8 lb.	6 hr.	\$ 4

The *quotient* found by dividing 6 by 2 is the *ratio* of 6 to 2. In general, the *ratio* of a number to another is the quotient obtained by dividing. The ratios of 6 to 3, and of a to b, may be written in the form $\frac{6}{3}$, $\frac{a}{b}$, and read "6 over 3," "a over b," meaning 6 divided by 3, a divided by b. In arithmetic $\frac{6}{3}$ is usually read 6 thirds.

Exercises

Express the ratios of the following, reducing each to the lowest terms.

1. 42 to 56.

Solution: Ratio $= \frac{42}{56}$.

Dividing numerator and denominator first by 2 and then by 7, we have

$$\text{Ratio} = \frac{\overset{\overset{3}{\cancel{21}}}{\cancel{42}}}{\underset{\underset{4}{\cancel{28}}}{\cancel{56}}} = \frac{3}{4}.$$

2. 4 to 9.
3. 8 to 18.
4. 18 to 33.
5. 12 to 15.
6. 198 to 12.
7. 24 to 63.
8. 3 to a.
9. y to x.
10. d to t.
11. $\frac{7}{8}$ to $\frac{3}{4}$.
12. $\frac{2}{3}$ to $\frac{5}{6}$.
13. $\frac{36}{40}$ to $\frac{27}{35}$.

Reduce the following fractions:

14. $\frac{60}{90}$.
15. $\frac{33}{44}$.
16. $\frac{21}{28}$.
17. $\frac{72}{81}$.
18. $\frac{125}{100}$.
19. $\frac{35}{40}$.
20. $\frac{56}{124}$.
21. $\frac{76}{132}$.
22. $\frac{63}{846}$.

Measuring by Hundredths

18. Expressing length as hundredths of a unit. On squared paper draw a segment 10 cm. long (one decimeter). In the following this is to be used as a

unit of measurement (Fig. 41). Imagine each centimeter divided in 10 equal parts (millimeters) as on the centimeter ruler.

Each millimeter is $\frac{1}{10}$ of a centimeter, or $\frac{1}{100}$ of the unit ($\frac{1}{100}$ of a decimeter).

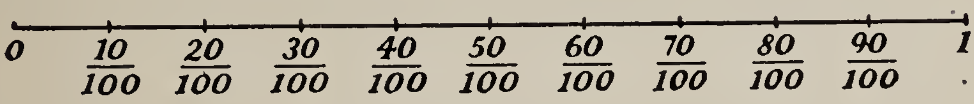


FIG. 41

Each centimeter is therefore $\frac{10}{100}$ of the unit.

We shall now see how to measure line-segments to the nearest one-hundredth part of the unit.

EXERCISES

1. Show from a drawing that a segment equal $\frac{1}{2}$ of a decimeter is equal to $\frac{50}{100}$ of it.

2. Using a decimeter as a unit, express the following line-segments as hundredths.

$\frac{1}{5}$ of a decimeter = ———— hundredths = $\frac{}{100}$.
$\frac{3}{4}$ of a decimeter = ———— hundredths = $\frac{}{100}$.
$\frac{4}{5}$ of a decimeter = ———— hundredths = $\frac{}{100}$.
$\frac{1}{3}$ of a decimeter = ———— hundredths = $\frac{}{100}$.

19. Meaning of per cent. In Exercise 2 hundredths were expressed by means of fractions having 100 as denominator. The word *per cent,* meaning hundredths, is also used. The sign %, read *per cent,* has the same meaning.

EXERCISES

1. The results of Exercise 2 may be expressed as follows:
$\frac{1}{5} = \frac{20}{100} =$ 20 hundredths = 20 per cent = 20%.
$\frac{3}{4} = \frac{75}{100} =$ 75 hundredths = 75 per cent = 75%, etc.
Express similarly the equivalents of $\frac{4}{5}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{8}$, $\frac{3}{8}$, $\frac{3}{5}$.

2. If part of a unit is 5%, or $\frac{5}{100}$, of the unit, what per cent is the remaining part?

3. Make drawings to show 10%; 25%; 50%; 75%.

4. Draw a line-segment of any length. Mark off 25% of it; 15% of it.

5. Express the following per cents as hundredths: 3%; 8%; 25%; 60%.

6. Make a drawing to show 10% of 2.

Draw a segment 2 decimeters long (Fig. 42). Since 10% of a decimeter is equal $\frac{10}{100}$ of it, show from the drawing that

$$\frac{10}{100} \text{ of } 1+\frac{10}{100} \text{ of } 1.$$
$$=\frac{10}{100} \text{ of } 2.$$

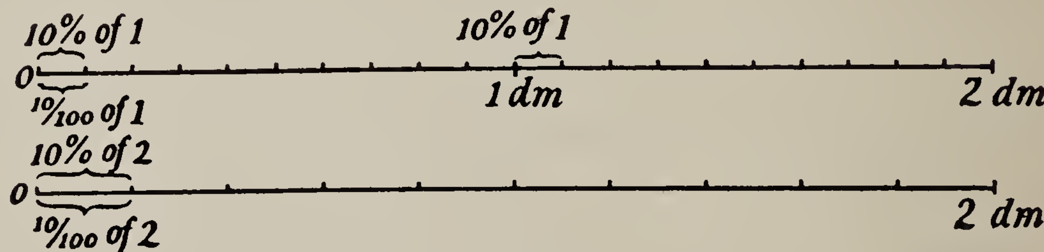


FIG. 42

7. Exercise 6 shows that 10% of 2 means $\frac{10}{100}$ of 2, or $\frac{10}{100}\times 2$. State the meaning of each of the following:

8% of 4; 6% of 8; 3% of 7.

8. Find 5% of 40.

Solution 1: 5% of 40 $=\frac{5}{100}$ of 40 $=\frac{5}{100}\times 40=\dfrac{\cancel{5}\times\overset{2}{\cancel{40}}}{\underset{\cancel{5}}{\cancel{100}}}=2.$

Solution 2: 5% of 40 $=.05\times 40=2$.

9. Find 5% of 60; 10% of 80; 25% of 200.

20. Uses of per cent in daily life. Measurement by hundredths is used widely in business, in scientific work, and in statistical reports as shown by the following statements:

A man saves 10% of his salary.

A merchant makes a profit of 20% on his sales.
The price of a suit is reduced 15%.
Milk tests 4% butter fat.
He is 100% American.
The bank pays 3% for the use of money.
Tell what each of these statements means.

21. What every pupil should know and be able to do. In the preceding pages the meaning of line-segment has been made clear through measuring and drawing.

Below is a list of terms and facts which every pupil should understand.

1. The meaning and correct use of the terms: Line, point, segment, unit-segment, ratio of two segments, scale drawing, metric system, exact and approximate measure, literal number, ratio of numbers.

2. The meaning of the following principles:

a. Through one point any number of straight lines may be drawn.

b. Only one straight line can be drawn through two points.

c. Two intersecting straight lines can have only one point in common.

3. Understanding of what *per cent* means.

4. The tables of linear measure in the English and metric systems.

22. Typical problems and exercises. The pupil should be able to use the ruler, squared paper, and compass in measuring and drawing line-segments; to

add, subtract, multiply, and divide accurately common and decimal fractions; and to write a satisfactory test paper on questions and problems of the type given below.

1. Draw a line-segment and find the length using only a ruler; using compass and ruler; using compass and squared paper.

2. Draw a segment and measure it to three figures; to the nearest sixteenth of an inch.

3. Explain the metric system of measuring lengths. How are meters changed to centimeters? Centimeters to decimeters?

4. State the meaning of the following symbols: $=$, $>$, $<$.

5. Add and subtract as indicated: $18\frac{1}{2}+10\frac{1}{4}-15\frac{3}{8}$.

6. Draw a line-segment. Mark off 15% of it.

7. Multiply $3\frac{1}{2}\times 2\frac{4}{7}\times 8\frac{2}{3}$.

8. Find the ratio $\frac{2.38}{3.15}$ to three figures.

9. Find the average of the following measures:
3.64, 3.59, 3.60, 3.61, 3.63, 3.62.

10. The dimensions of a room are 24 ft. by 18 ft. Make a drawing of the room representing 10 ft. by one inch.

11. Reduce the ratio $\frac{210}{120}$ to lowest terms.

12. Write a paper on one of the following topics:
a. The need of measurement in daily life.
b. How standard units of measurement have come into use.
c. Scale drawings.
d. The metric system.

CHAPTER II

HOW WE USE LINE-SEGMENTS IN PICTURING NUMERICAL FACTS

Representing Numerical Facts by Graphs and Tables

23. Picture-representation of numerical facts. Newspapers and magazines often represent numerical statistics, scientific data, or facts by the use of drawings and pictures. Such pictures are known as *pictograms*, or *picture graphs*. Thus, the picture graph (Fig. 43) illustrates the amounts of the leading crops in the United States in 1917, each being pictured by the number of bags on the truck. The differences in the crops are easily seen by comparing the various loads as to size.

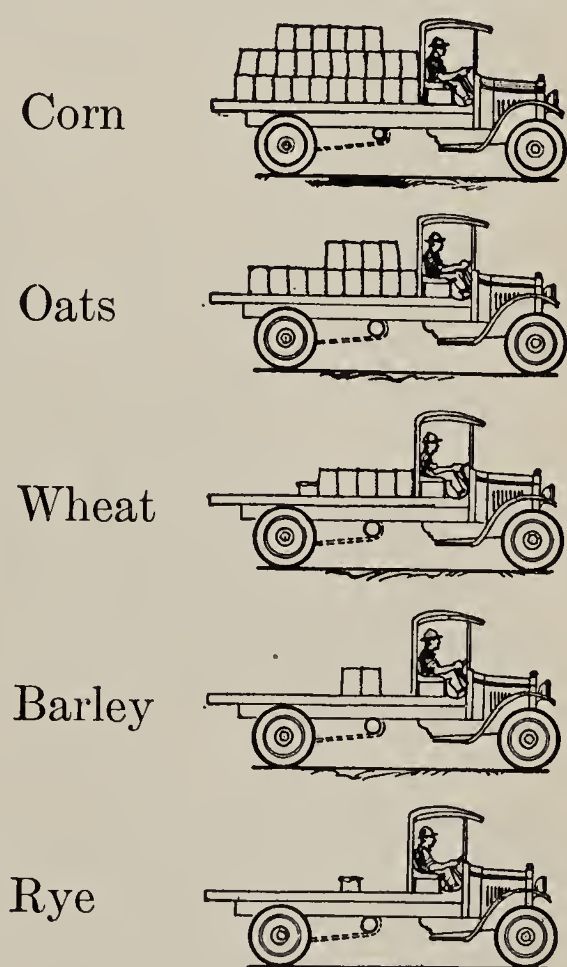


Fig. 43

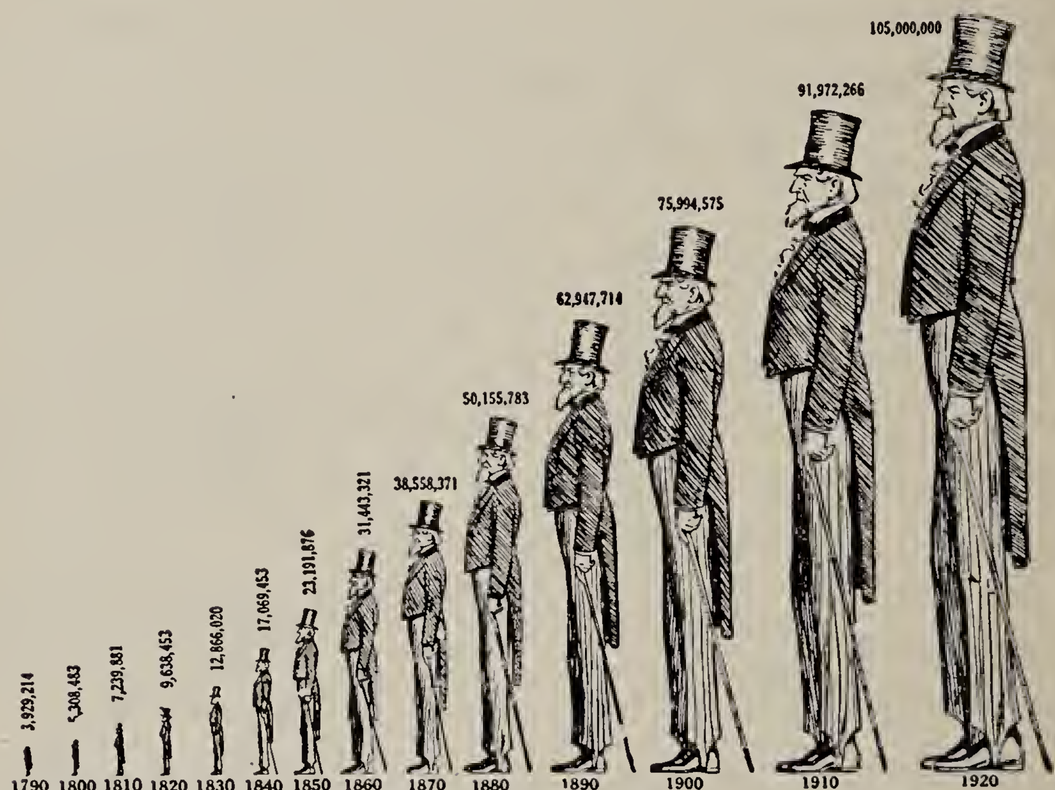
The same facts might be stated in the form of the table on page 36.

REPORT OF THE UNITED STATES DEPARTMENT OF AGRICULTURE FOR 1917

Corn	*Oats*	*Wheat*	*Barley*	*Rye*
3,159,494,000	1,587,286,000	650,828,000	208,975,000	60,145,000

The table should be read as follows: In 1917 the United States raised 3,159,494,000 bushels of corn, etc.

In the picture graph (Fig. 44) the *height* of each figure represents a number of millions. It shows how



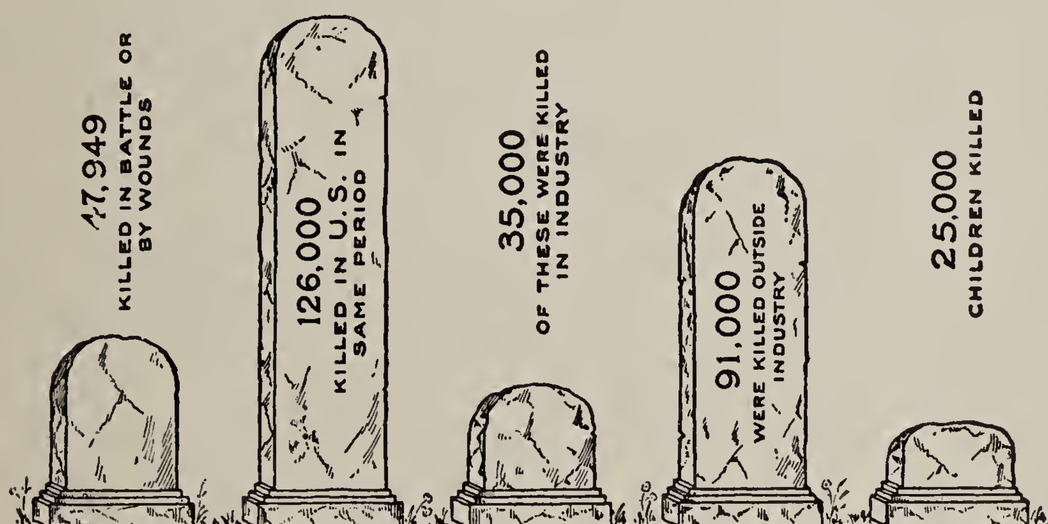
Courtesy of the Literary Digest, N. Y.

FIG. 44

our continental population kept up a steady rate of increase from one decade to the next. However, in 1920 the population was only 105 millions, while it would have been about 111 millions if it had maintained the rate of increase of the previous decades.

In some cases when the picture graph is used to represent numerical facts, there is a danger of forming wrong impressions. In comparing data represented by two-dimensional figures when only length is to be considered, the surfaces covered may not have the same relations to one another as the lengths. Thus, the *surface* of the 1920 figure is 4 times as great as that of the 1880 figure, while the *length* of the first is only 2 times as great as that of the second. Hence, a comparison of the surfaces would lead us to believe that the population of 1920 was 4 times as great as that of 1880, which is not in keeping with the fact.

In Fig. 45 we find a pictogram illustrating the fatalities at home while this country was at war.



Courtesy of the Literary Digest, N. Y.

FIG. 45

The diagram (Fig. 45) is not open to the objection raised against Fig. 44, for all the stones have the same width, which makes it possible to compare the surfaces in the same way as the heights. In Fig. 45, not only the heights of the stones but also the surfaces represent the number killed.

The pictorial method is preferred to the tabular method mainly because the meaning of numerical facts is more easily seen and understood from pictures than from tables. From a row of figures it is not easy to grasp the real facts. For figures must be studied thoroughly, but a picture of numerical facts can be understood without difficulty. Pictures present to the mind in a definite, clear and comprehensive manner the relations between facts. It is therefore important that readers of magazines and newspapers learn how to interpret numerical facts and relations of facts when they are stated in graphical form.

Several other types of graphical representation may be used for certain kinds of facts. Some of them may not give an impression as clear as that gained by a picture graph, but they have other advantages, such as being easily made and being very accurate. These types are to be discussed in §§24 to 27.

24. How to represent numbers by segments. Draw a line-segment 3 centimeters long. This segment is said to *represent* the number 3. In the following exercises we shall learn how to represent numbers by segments.

EXERCISES

1. Draw a segment 5 cm. long. What number does this segment represent?

2. Draw a segment $3\frac{1}{2}$ in. long. What number is represented by the segment?

3. For each of the following numbers find the unit segment most convenient for representing the number by a segment. Then draw the segments representing the numbers

$$4.5,\ 6,\ 2\tfrac{5}{8},\ 1\tfrac{3}{4},\ 3\tfrac{4}{5}.$$

4. Represent the number 2.38 by a segment.

Directions: As unit select a segment equal to 2 cm.

Draw a segment of convenient length, as AB (Fig. 46).

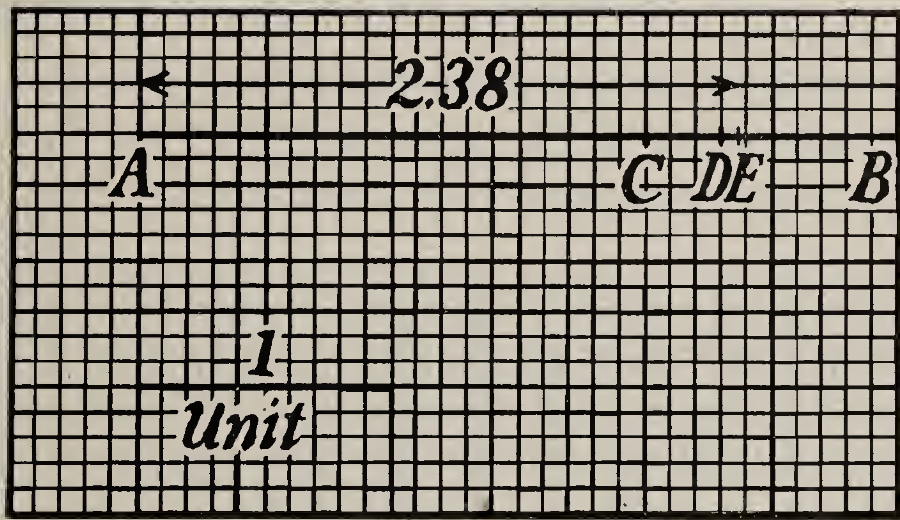


Fig. 46

On AB lay off the unit twice, extending from A to C. AC is then equal to 2 and represents the first digit of 2.38.

To represent the second digit, lay off from C in the direction CB a segment CD three times as long as the side of a small square. The length of CD is .3.

Finally, starting at D lay off DE in the direction DB by *estimating* .3 of the side of a small square.

Show that AE represents the number 2.38.

5. Represent, as in Exercise 4, the following numbers by segments: 1.55, 2.25, 1.37, 1.42.

25. What graphical representation means. When a segment is used to represent a number it is called a *graph* of the number, and it is said to represent the number *graphically*.

26. How to make bar graphs. The table below states the number of hours electric light is used in the average residence in a certain city for each month of one year.

Months	*Jul.*	*Aug.*	*Sep.*	*Oct.*	*Nov.*	*Dec.*	*Jan.*	*Feb.*	*Mar.*	*Apr.*	*May*	*June*
Average daily hours	1.7	2.2	3	5	6.1	6.8	6.5	5.3	4.2	2.5	1.9	1.5

The table should be read as follows: In July the average family burns electric light for 1.7 hours, etc.

To represent these facts graphically, the months in the table may be marked off horizontally (Fig. 47) on

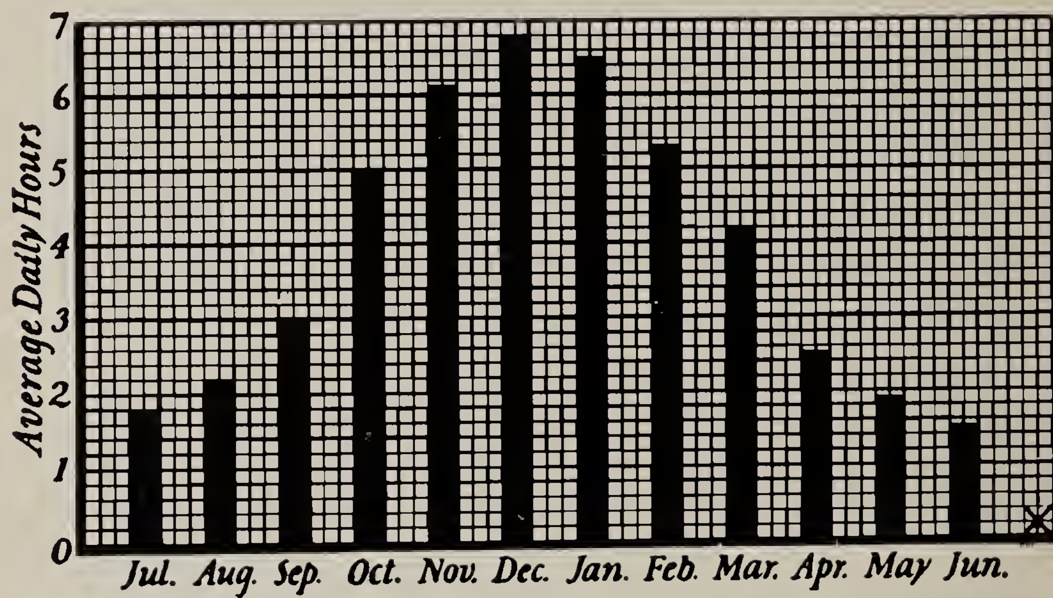


Fig. 47. Bar Diagram Showing the Average Daily Hours Electric Light Is Used in the Average Residence.

the segment OX, and the number of hours may be represented by vertical segments drawn upward from the points on OX which represents the various months.

This method of stating facts graphically not only supplements the tabular representation but has some advantages over it. For example, all the facts given in the table above can be seen almost with one glance at the diagram.

Furthermore, the graph shows clearly how much more light is used in December than during any other month; that in June the amount of light used is least; that in winter light is used, approximately four times

as many hours as in summer. The graph illustrates how the amount of light used depends upon the time of the year, and explains how our electric light bills *vary* (change) from one month to the next.

The type of graph used in Fig. 47 is called a **bar graph.** The bar graph is easily constructed and understood. The *length* of the bar shows the *size* of the number.

Exercises

1. Figs. 48 and 49 are bar graphs. The first line (Fig. 48) means that in 1910 in China the population was 302 million. Explain the meaning of each of the others. Make comparisons for the different countries by means of ratios.

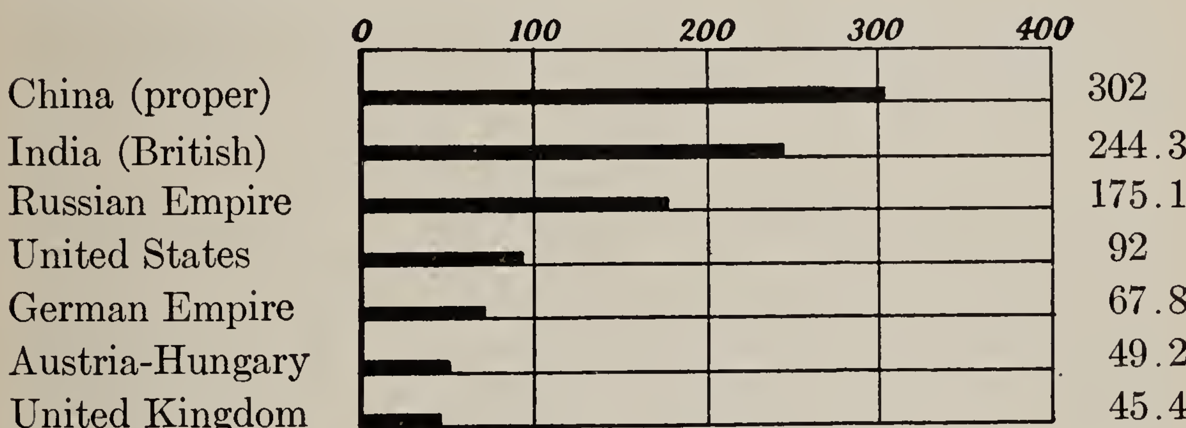


Fig. 48. Bar Diagram Showing Population, in Millions, of Leading Countries in 1910.

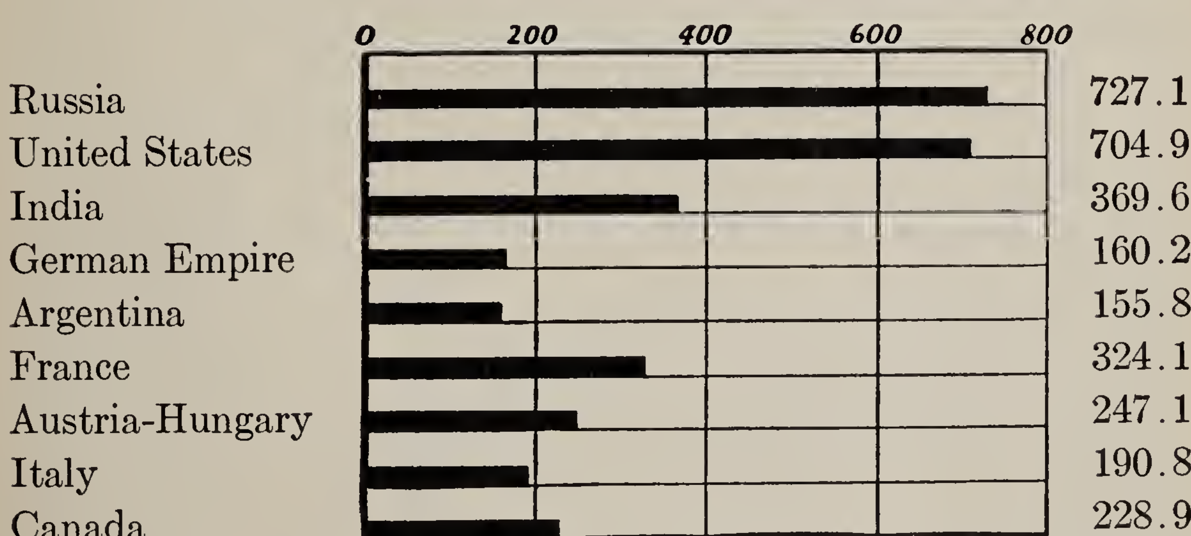


Fig. 49. Bar Diagram Showing the World Wheat Crop in Millions of Bushels. Average for 1911–1913.

2. As in Exercise 1, explain the meaning of Fig. 50 and make comparisons.

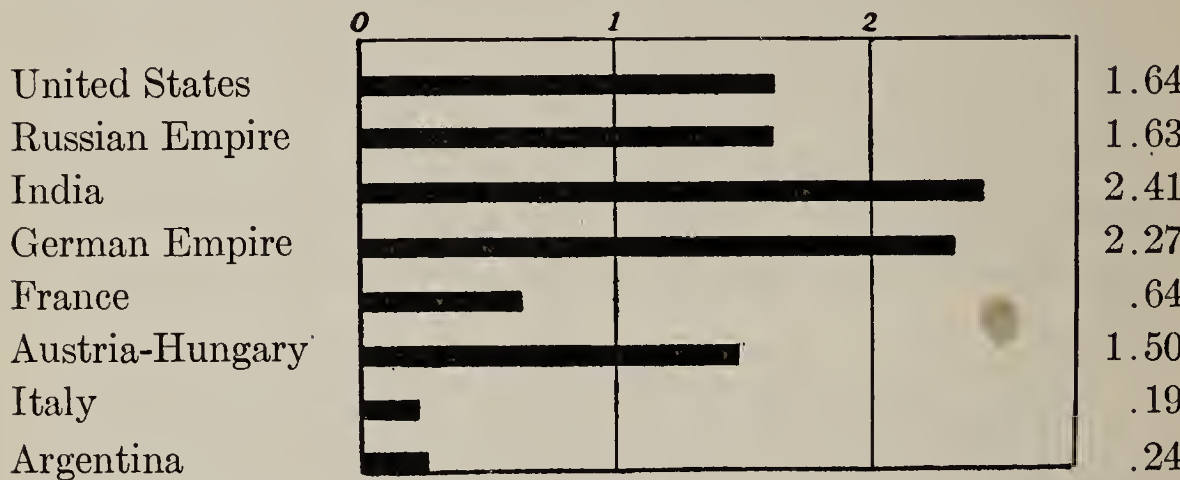


Fig. 50. Diagram Showing World Sugar In Millions of Tons for 1911–1913.

3. Using 1,000,000 as a unit make a bar graph to illustrate the areas of the continents given in the table below.

Areas of the Six Continents

Continents	*Areas*	*Continents*	*Areas*
North America...	8,000,000	Asia............	17,000,000
South America....	6,850,000	Africa...........	11,000,000
Europe..........	3,800,000	Australia........	3,000,000

4. Represent by means of a graph the following table:

Exports During the Years 1914

Cotton....................................	610,000,000
Foodstuffs................................	430,000,000
Iron and Steel............................	300,000,000
Mineral Oil...............................	152,000,000
Copper Manufactures.......................	146,000,000
Wood and Manufactures.....................	103,000,000
Tobacco and Manufactures..................	61,000,000
Coal......................................	60,000,000
Carriages.................................	52,000,000
Cotton Goods..............................	50,000,000

5. In the adjoined graph (Fig. 51) the whole bars represent income from sales; the white parts represent profits; the shaded parts represent costs. Give the approximate cost, profit, and sale for each year.

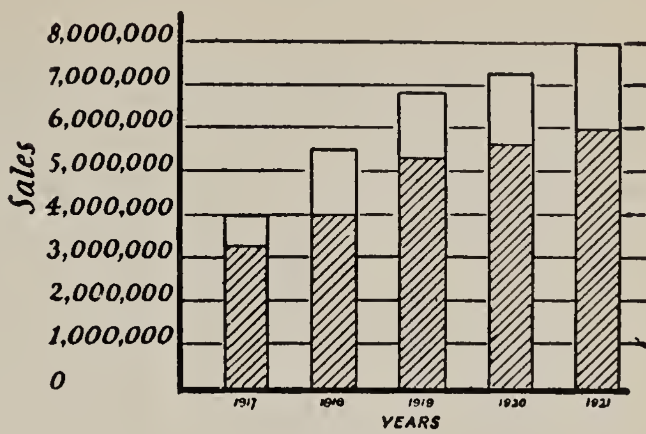


Fig. 51

Using 1,000,000 as a unit, *i.e.*, using only the figures indicating millions, draw a bar graph for each of Tables 6 to 9 below.

Some of the Principal Crops of the World. Average Number of Bushels in 1911–1913

Countries	6. *Barley*	7. *Potatoes*	8. *Rye*
United States.........	187,417,700	348,303,000	36,721,300
Russian Empire.......	484,848,000	1,287,880,700	935,010,300
India.................	38,097,700	....	....
German Empire.......	157,921,700	1,698,826,000	455,181,700
France...............	47,608,700	506,884,700	48,078,700
Austria-Hungary......	153,437,000	642,149,000	163,640,000
Italy.................	10,029,300	61,410,300	5,390,300
United Kingdom.......	62,528,300	259,482,700	1,666,700
Argentina............	5,096,700	38,029,000	1,748,300
Canada...............	47,370,700	78,222,300	2,406,700
Australia............	2,816,700	13,842,700	95,700

9. World Corn Crop in Millions of Bushels. Average for 1911–1913.

United States	2701
Russian Empire	78.1
India	87.6
France	20.6
Austria-Hungary	210.8
Italy	100.2
Argentina	251.9
Canada	17.6
Australia	10.4

10. In the table below the "foreign white stock" in the United States in 1910 is the totality of the white population which is foreign either by birth or by parentage, thus including only the first and second generation. Using 100,000 as a unit, make a bar graph of the numbers in the table, as 10.0, 8.8, 2.1, 1.7, etc.

Mother Tongue.	*Number*	*Per cent.*
English and Celtic*	10,037,420	31.1
German	8,817,271	27.3
Italian	2,151,422	6.7
Polish	1,707,640	5.3
Yiddish and Hebrew	1,676,762	5.2
Swedish	1,445,869	4.5
French	1,357,169	4.2
Norwegian	1,009,854	3.1
Total eight mother tongues	28,203,407	87.5
Other mother tongues	4,039,975	12.5
All mother tongues	32,243,382	100.0

*Includes persons reporting Irish, Scotch or Welsh.

11. A grocery order retailing for $22.00 in 1917 sold for $24.50 in 1918, for $36.00 in 1919, for $50.00 in 1920, and for $18.50 in 1921. Make a bar graph of these facts, showing how food prices changed from 1917 to 1921.

27. How to make line graphs. The diagram (Fig. 52) pictures the temperature readings taken each hour on a certain day. By joining the top points by a continuous line the changes in temperature are exhibited in a more satisfactory way than they would be if the bars alone were used. In fact, if the bars were omitted entirely, leaving only the line passing through the top points, this line would picture the changes in temperature and the hourly readings.

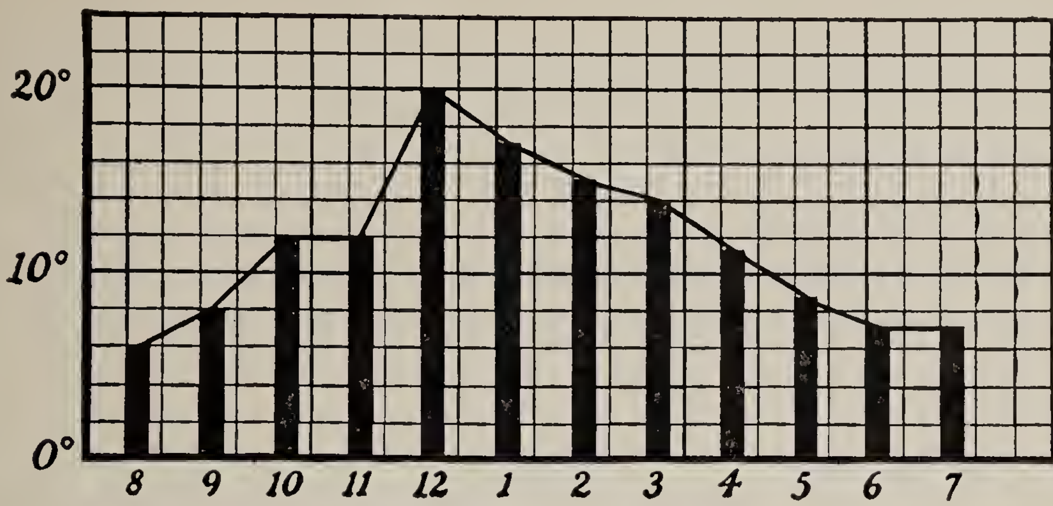


Fig. 52

The average heights of boys and girls, given in the table below, are represented graphically in Fig. 53. The *vertical* segments are omitted in the graph and only the top-points are marked.

For when squared paper is used the vertical lines on the paper make the heavy bars unnecessary.

Ages	*Boys*	*Girls*
2 years	1.6 ft.	1.6 ft.
4	2.6	2.6
6	3.0	3.0
8	3.5	3.5
10	4.0	3.9
12	4.8	4.5
14	5.2	4.8
16	5.5	5.2
18	5.6	5.3
20	5.7	5.4

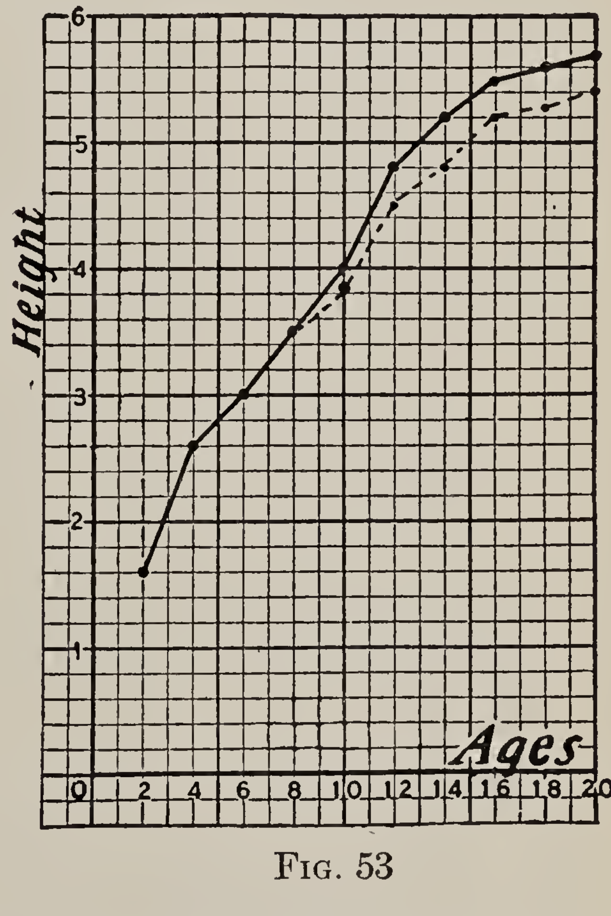


Fig. 53

By joining the top-points with lines the changes in growth may be brought out even better than by

drawing only the isolated facts given in the table. This type of graph is called a **line graph.**

State some facts shown more clearly in the graph than in the table.

The following table of statistics gives the population of the United States from 1800 to 1910 and is represented graphically in Fig. 54.

Year	*Population*
1800	5,308,483
1810	7,239,881
1820	9,638,463
1830	12,860,702
1840	17,063,353
1850	23,191,876
1860	31,443,321
1870	38,558,371
1880	50,155,783
1890	62,947,714
1900	75,994,575
1910	91,972,266

The graph in Fig. 54 represents the same facts. It is a combination of the bar graph and line graph.

Compare this graph with the pictogram in Fig. 44.

The two types of graphs shown in §§26 and 27 are used for different purposes. The *bar graph* should be used to picture *unrelated* facts, as the production of a certain article by various nations, the monthly number of automobile accidents, the wealth of several countries. The *line graph* not only

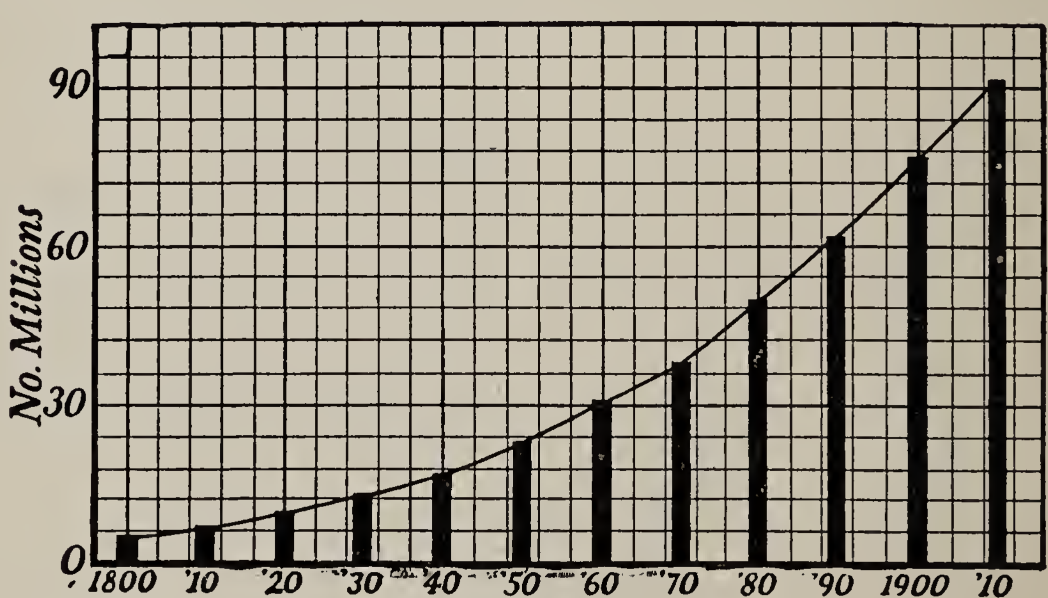


FIG. 54

illustrates given facts and enables us to make comparisons, but makes it possible to derive additional facts that are not actually stated in the table, or drawn in the diagram. For example, in Fig. 54, we may estimate the population in 1895, or in 1855; we may even predict what under ordinary conditions should be the population in 1920. *Changing* prices of a commodity, temperatures, stock fluctuations, cost relations, are best pictured by *line* graphs.

Exercises

1. The graph (Fig. 55) shows the hourly variation of telephone calls in a great city. Study the graph and answer the following

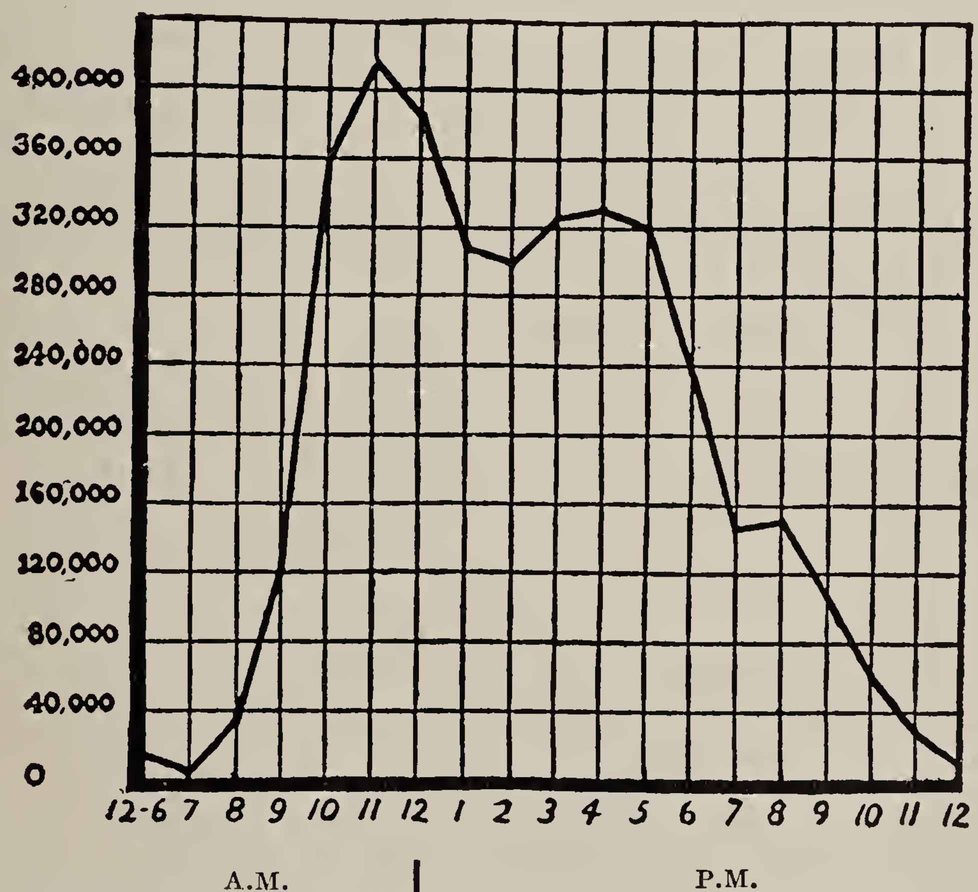


Fig. 55. One Day of New York's Telephone Traffic.

questions. When do people use the telephone most often? How do the calls between nine and ten compare with the calls between eight and nine? How do you account for the drop between twelve and two?

2. The graph (Fig. 56) gives the changes in the price of wheat for one year. What was the price on the first day of each month? When was it lowest? When was it highest? What is your explana-

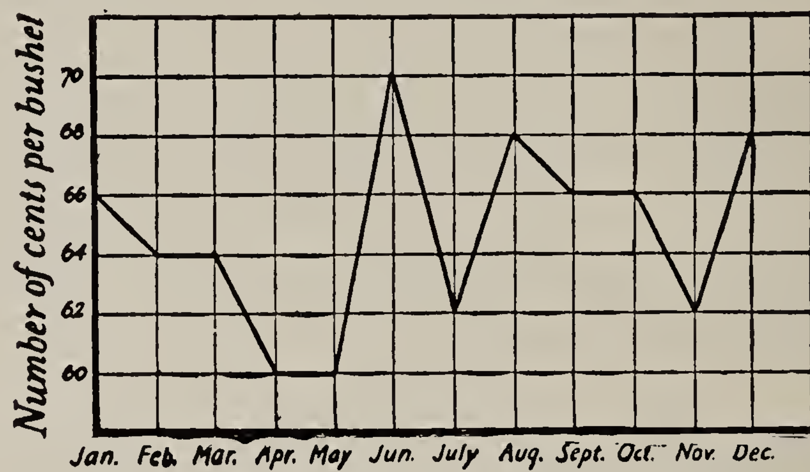


Fig. 56

tion of the last two answers? In which month was the advance greatest? When was the drop greatest? When was the price 65 cents?

3. Study the graph (Fig. 57). Find out why immigration was low in 1862, 1896 and high in 1883, 1907. How do you explain the drop after 1907?

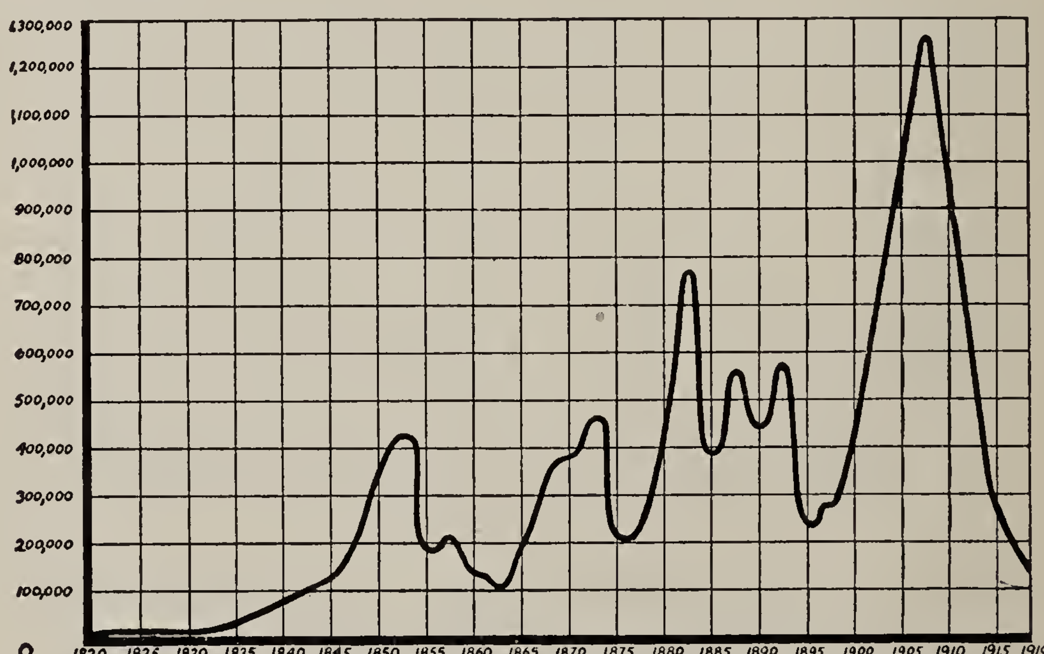


Fig. 57. The Rise and Fall of Immigration.

4. State some advantages of the graphical over the tabular representation of facts.

5. The average monthly rainfall or snowfall for a certain city is given in the table below. Make a line graph representing this table letting 2 cm. denote 1 inch. Follow the directions given on page 40.

Months	*Jan.*	*Feb.*	*Mar.*	*Apr.*	*May*	*June*	*July*	*Aug.*	*Sep.*	*Oct.*	*Nov.*	*Dec.*
Inches	2.8	2.30	2.56	2.70	3.59	3.79	3.61	2.83	2.91	2.63	2.66	2.71

6. The hourly temperatures on a certain day were as follows:

Time of day	*6:00 a.m.*	*7:00*	*8:00*	*9:00*	*10:00*	*11:00*	*12:00*	*1:00*	*2:00*	*3:00*	*4:00*	*5:00*
Temperatures	48°	49°	53°	55°	61°	65°	70°	72°	72°	70°	65°	63°

Make a graph of this table and tell what the graph shows. Find the average temperature.

7. Make a graph of the number of pupils making a certain number of points in a written test recorded below.

No. of points	*12*	*11*	*9*	*8*	*7 or less*
No. of pupils	3	7	9	4	2

8. The average length of day from sunrise to sunset varies (changes) with the latitude. Represent by a line graph the following average lengths of days in latitude 45°, letting 1 cm. represent one hour.

Months	*Jan.*	*Feb.*	*Mar.*	*Apr.*	*May*	*June*	*July*	*Aug.*	*Sep.*	*Oct.*	*Nov.*	*Dec.*
Hours	9.1	10.4	11.9	13.5	14.9	15.6	15.3	14.1	12.6	11.1	9.6	8.8

9. Represent by a bar graph the fire losses in the city of Chicago given in the following table:

Years	*1912*	*1913*	*1914*	*1915*	*1916*	*1917*	*1918*
Number of millions of dollars lost	6.65	5.01	6.01	4.9	5.49	4.18	4.08

10. The table below gives the death rates caused by influenza and pneumonia in some of the large cities during the epidemic in 1918. Compute the increase in the death rate from 1917 to 1918. Make a graph of the increase and tell what the graph shows.

Death Rates Caused by Influenza and Pneumonia

Cities	*1917*	*1918*	*Increase*
Cleveland........................	13.9	16.0	2.1
Chicago........................	14.0	17.1	....
New York........................	15.2	18.8	....
Dayton........................	15.9	19.6	....
Los Angeles........................	12.5	16.4	....
Cincinnati........................	15.5	20.6	....
Louisville........................	16.3	21.0	....
Richmond........................	18.5	23.6	....
San Francisco........................	15.0	20.5	....
Boston........................	15.4	22.0	....
New Orleans........................	19.9	25.9	....
Philadelphia........................	17.1	24.2	....
Pittsburgh........................	18.2	25.4	....

11. Select the most convenient unit and then make a bar graph representing the facts in the table below which gives the production of corn in several states.

Suggestion: Use only the first three figures for making the graph.

Compare by means of graph, and by means of ratio, the productions of the following states: Iowa with Missouri; Iowa with Indiana; Missouri with Kentucky; each state with Illinois.

Production of Corn in Several States

States	*Number of Bushels*	*States*	*Number of Bushels*
Illinois...........	444,843,000	Indiana........	208,522,000
Iowa............	411,656,000	Ohio...........	162,859,000
Missouri.........	263,463,000	Kentucky......	126,859,000

12. Make graphs representing the heights of the following buildings in New York City: Woolworth 750 feet, Metropolitan 700 feet, Singer 612 feet, Equitable 486 feet, Times 420 feet, Flatiron 286 feet. Represent by a bar the tallest building in your city. Find the ratio of each of the buildings above to the height of your tallest building.

13. The height of the average man is about 5 ft. 5 inches. The average weight for men 5 ft. tall is given in the following table for different ages. Make the graph.

Ages	*15–24*	*25–29*	*30–34*	*35–39*	*40–44*	*45–49*	*50–54*	*55–59*	*60–64*	*65–69*
Pounds	134	138	141	143	146	147	149	149	148	147

14. Represent graphically the weights of boys and girls given in the following table.

Ages	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*
Weight of boys	45	49	54	59	65	70	76	84	95	107	111
Weight of girls	43	47	52	57	62	69	78	88	98	106	112

15. The table below gives a record of a typical day's demand for electricity in New York. Make a graph of this table and explain the following facts shown in the graph: very little demand at 4 A.M.; the rapid increase from 5 to 7 A.M., and from 7 to 9 A.M.; the great demand at 11 A.M.; the sudden drop at 12:30; the largest demand at 5 P.M.; the slow decrease from 7 to 9 as compared with that from 5 to 7 and from 9 to 12 P.M.

A.M.

Time of day	*12*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*
1000 kilowatts	65	53	42	39	38	41	58	75	105	147	165	168	165

P.M.

Time of day	*12:30*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*
1000 kilowatts	135	158	187	180	180	233	198	159	154	135	118	90	72

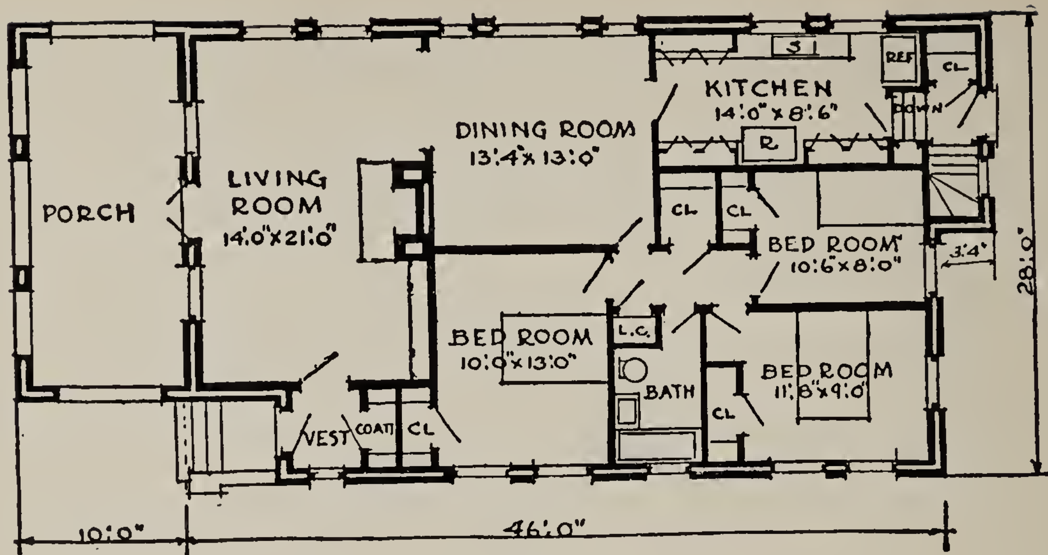


Fig. 58

16. Make a scale drawing of the floor plan (Fig. 58).

17. A department store reports the sales and profits for a period of five years as follows:

Year	*1918*	*1919*	*1920*	*1921*	*1922*
Sales in millions	4	$5\frac{1}{2}$	6.9	7.1	8
Profits in millions	.7	$1\frac{1}{2}$	1.6	1.6	2

Represent the table graphically.

18. On a normal business day the average number of passengers carried in and out of Chicago by the suburban trains of the Chicago and North Western Railway is approximately 60,000.

The number of trains necessary to handle this business is 166.

On August 1st, the first day of the street car strike, the suburban trains of this company carried 109,810 passengers.

The number of trains required for this business was 225.

On August 2nd the number of passengers carried was 134,704. The number of trains operated, 245.

On August 3rd the number of passengers carried was 135,619. The number of trains operated, 243.

On August 4th the number of passengers carried was 134,822. The number of trains operated, 242.

Make tabulated statements of these facts as follows:

	No. Passengers	No. Trains	Passengers Per Train
Normal	60,000	166	361
August 1st			
August 2nd			
August 3rd			
August 4th			

Represent by line graphs the passengers' increase and the train increase. Represent by a bar graph the number of passengers per train.

19. The following table shows the loss from absences in a high school for one year. Represent the facts in the table graphically.

Months	*Oct.*	*Nov.*	*Dec.*	*Jan.*	*Feb.*	*Mar.*	*Apr.*	*May*	*June 10 days*	*Year*
No. of different pupils absent	94	114	129	299	259	188	205	175	71	
Aggregate days absent	241	281	260	652	440	624	633	404	161	3,696
Per cent of daily absences	2.1	2.5	3.3	5.4	4.7	7.1	5.8	3.5	3.1	4.34

28. What every pupil should know and be able to do. Having made a study of this chapter the pupil should understand how numerical facts may be represented arithmetically as in tables, or geometrically as in graphs. He should be able to tell what a given bar graph and line graph shows, and he should know how to make a graph from a given table of numerical facts.

29. Typical exercises. One who understands this second chapter should be able to work the following problems:

1. Draw a bar graph representing these numbers: 2.50, 3.41, 4.23, 1.68.

2. Some of the longest rivers we read about in the study of geography of the United States expressed in thousands of miles are approximately of the following lengths.

Missouri-Mississippi 4.2, Yukon 2.2, St. Lawrence 2.1, Arkansas 2, Rio Grande 1.8, Columbia 1.4, Colorado 1.4. Make a bar graph representing these approximate lengths.

3. The temperature record for a certain day reads as follows:

Hour..........	*6 a.m.*	*7*	*8*	*9*	*10*	*11*	*12*	*1 p.m.*	*2*	*3*	*4*	*5*	*6*
Temperature....	16°	18°	19°	19°	21°	22°	24°	24°	20°	18°	18°	16°	12°

Make a line-graph showing the changes in temperature on that day.

4. Write a paper on the meaning and uses of graphs.

CHAPTER III

REPRESENTING NUMERICAL FACTS BY FORMULAS

How to Make a Formula

30. A third method of representing numerical facts. In chapters I and II we have studied two methods of representing numerical facts, the *arithmetical* by which facts were arranged in the form of tables, and the *geometrical* by which they were represented in geometric figures, *e.g.*, in graphs. There is a third method, which we shall study in this chapter and which is the most effective of the three. It represents numbers by means of letters. It is used not only in mathematics but in science, shop work, and engineering. The following exercises introduce this new method:

Exercises

1. On a sheet of centimeter squared paper lay the edge of a ruler, which is divided in inches, along one of the heavy lines. Using the centimeter as unit, measure carefully segments equal to 1 in., 2 in., ..., 8 inches and tabulate the results as follows:

Number of Inches	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Number of centimeters	2.54	5.08						
Ratios								

Find the ratio of each number in the second row to the corresponding number in the first row. If this is done accurately, you

will see that the number of centimeters in a segment is approximately 2.54 times the number of inches.

The equation $c=2.54i$, stated in words, means the *number of centimeters* is 2.54 times *the number of inches.*

2. The cost of oranges of a certain size is 30c a dozen.

a. Complete the following table which gives the cost of oranges from 1 to 10 dozen.

Number of Dozen	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Cost in cents..........	30	60	...	...	...	...	...	...	...	....

b. Make a line graph illustrating the facts given in the table above. To represent the cost, let one side of a large square on the graphing paper represent 30.

c. From the graph determine the cost of $4\frac{1}{2}$ dz.; $7\frac{1}{2}$ dz.; $8\frac{1}{2}$ dozen.

d. From the graph determine how many oranges can be bought for $0.60, $1.80, $2.10, $2.70.

e. In the table above compare by means of ratios the numbers in the second row with the corresponding numbers in the first row. What relation do you find between cost and the number of dozens bought?

f. Express the results of (*e*) in one general statement, giving the cost in terms of the number of dozens.

g. Let the number n represent the *number of dozens,* and the number c the *cost* in cents. Show that the statement in (*f*) may be expressed briefly in the form $c=30\times n$.

h. The statement $c=30\times n$ in words means: the number of cents is equal to 30 times the number of dozens.

i. When $n=1$, show that $c=30\times1=30$.

j. Let $n=2$, and show that $c=60$.

k. Show how to obtain all of the facts stated in the table and in the graph from the brief statement $c=30\times n$.

31. What is meant by the value of a literal number. In statements like $c=30\times n$, the literal number n may stand for *any* number, as, 1, 2, 3, etc. The

literal number c, then stands for the corresponding numbers 30, 60, 90, etc. A number for which a literal number stands is a *value* of the literal number.

32. The meaning of the word formula. The statement $c=30\times n$ expresses the equality of the two numbers c and $30\times n$, and is therefore an *equation*.

Exercise 2, k (§30), shows that the equation $c=30\times n$ enables us to determine the cost of any given number of dozens of oranges selling at 30 cents a dozen.

When an equation is used to state briefly a rule for obtaining numerical facts from other related facts it is a *formula*.

Thus, $c=30\times n$ is a *formula*.

The following exercises teach how to "make formulas":

Exercises

1. *a.* Let n represent the number of articles purchased, let p be the price of each, in cents, and let c be the total cost in cents. In the table to the right find the values of c for given values of n and p. Thus, when $n=1$, $p=10$, $c=1\times 10=10$, etc.

n	p	c
1	10	$1\times 10=10$
2	8	
3	16	
4	4	
5	18	
10	p	
n	6	
n	p	

b. Make a formula for finding the cost c for n articles each sold at p cents.

c. Show that all the facts found in Exercise 1, *a*, can be obtained from the formula $c=n\times p$.

2. *a.* Make a table giving the number of inches, i, corresponding to the number of feet, f, letting f take the values 1, 2, 3, ...

b. Make a statement expressing in words, the number of inches in terms of the number of feet. Translating this statement into symbols, make a formula for finding the values of i, exercise 2, *a*, corresponding to values of f.

c. Using the facts given in the table of Exercise 2, *a*, make a graph of the formula $i = 12 \times f$.

3. *a*. A cubic foot of water weighs 62.5 pounds. Show that the number of pounds (weight) w, of a volume of v cubic feet is given by the formula $w = 62.5 \times v$. Translate this formula into words.

b. Tabulate corresponding values of v and w, exercise (*a*) and make a graph of the equation $w = 62.5 \times v$.

4. *a*. A train traveling over equal distances in equal time-spaces is said to have *uniform motion*. The distance traveled in the unit of time is the *velocity* (speed, or rate) of the train.

Complete the table at the right, stating values of the distances, d, corresponding to values of the time, t, for a train having a velocity of 20 miles an hour.

t	r	d
1	20	$1 \times 20 = 20$
2	20	
2.5	20	
$3\frac{1}{3}$	20	
4.75	20	
5	20	
6	20	
t	20	

b. In the table below find the values of d corresponding to given values of r and t.

c. Write a formula from which to find the distance, d, of a train traveling at a rate, r, for t hours.

t	r	d
1	30	$1 \times 30 = 30$
6	15	
12	40	
8	20	
15	10	
t	16	
18	r	
t	r	

5. Make a formula for finding the number of feet, f, from a number of yards, y. Translate this formula into words.

6. A boy earns d dollars a week for n weeks. State a formula for finding his total earnings t.

7. If n oranges cost C cents, state a formula for finding the cost, c, of one orange.

8. Make a formula for finding the number of gallons n, of oil, which pass through a pipe in m minutes, at the rate of 2 gallons a minute.

9. A river flows at the rate of r mi. an hour. Find the distance, d, an object will float in t hours.

10. Make a formula for finding the quotient, q, when the dividend is D and the divisor d.

11. A boy is n years old. What was his age, a, 5 years ago? What will be his age, A, 5 years from now?

12. A girl saves $2 a week for n weeks. Make a formula for finding her total savings x.

13. A girl receives 10 cents each time she does the shopping for her mother. Make a formula for finding the amount, t, which she earns in n days, if she goes shopping twice a day.

33. Various uses of the formula. As we go on in the study of mathematics we shall meet formulas frequently because the formula saves time and effort in the solution of many problems. Formulas are used in other school subjects. Thus, in science the formula helps us to find the distance a particle falls in a given time, and in physics many laws are stated as formulas. The formula plays an important part in every-day life. The automobile owner determines the horsepower of his gasoline engine by means of a formula, the business man may use it to compute interest on a sum of money, and the machinist to find the length of belting connecting two pulleys. Scientists, engineers, machinists, surveyors, insurance men, and many others use formulas, and should know the correct way in which to work with them. The importance of the formula is one of many reasons for studying mathematics.

Perimeter Formulas

34. Polygons. On notebook paper mark points A, B, C, D, E, and F, as shown in the diagram (Fig. 59) and join them with line-segments. The figure thus formed is called a **polygon.** The word *polygon* comes from the Greek and means a figure with *many angles.*

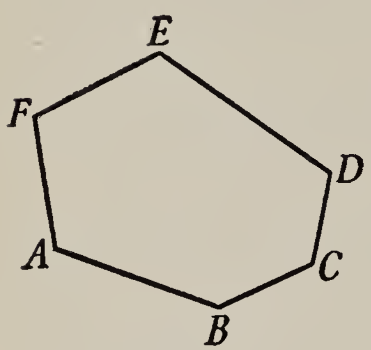


Fig. 59

What is the least number of segments needed to form a polygon?

The points A, B, C, etc., are the *vertices*, the segments AB, BC, CD, etc., are the *sides* of the polygon.

Measure the sides and find the sum

$$AB+BC+CD+DE+EF+FA.$$

The sum of the sides of a polygon is the *distance around*, or the *perimeter* of the polygon.

A polygon is called a *triangle, quadrilateral, pentagon, hexagon,* . . . . according as it has 3, 4, 5, 6, ... sides.

A polygon is *equilateral* when all the sides are of *equal length.*

Exercises

1. Measure each side of triangle ABC (Fig. 60) and find the perimeter by adding the lengths.

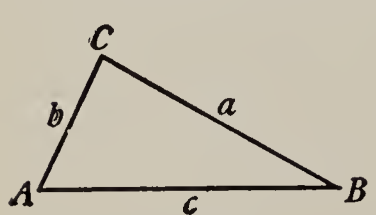


Fig. 60

Denoting the perimeter by p, state the result in the form of an equation.

When the sides are not measured, the perimeter, p, of triangle ABC is given by the formula $p=a+b+c$.

2. Without measuring the sides find the perimeter of the equilateral triangle (Fig. 61) and state the result in the form of a formula.

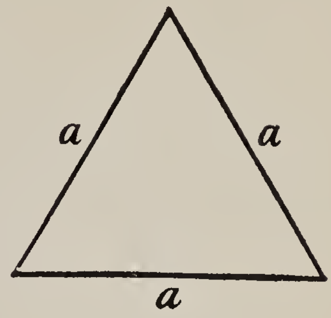


FIG. 61

3. What is the perimeter, p, of a triangle whose sides are $3x$ ft., $4x$ ft., and $5x$ feet? State the result in the form of an equation.

4. For each of the following equations sketch at least one polygon whose perimeter is expressed by the equation.

1. $p=3x$.	3. $p=6x$.	5. $p=10x$.
2. $p=5x$.	4. $p=8x$.	6. $p=12x$.

5. Find the perimeter of each of the polygons in Exercise 4 when $x=2.75$ cm.

6. Write the equations in Exercise 4 when the perimeter $p=148$.

7. The perimeter of an equilateral hexagon is 120 inches. Find the length of the side.

Solution: Let x be the number of inches contained in the side.

Show that $120=6x$.
Show that $x=\frac{1}{6}$ of $120=\frac{1}{6}\times 120$.
Hence, $x=20$.

8. The perimeter of an equilateral quadrilateral is 24. Find a side by means of an equation, as shown in Exercise 7.

9. The perimeter of an equilateral pentagon is 60. Find the side.

10. The perimeter of an equilateral decagon (10-side) is 28. Find the side.

35. Symbol for hence and therefore. The symbol used to denote *hence* or *therefore* is $\therefore$.

Addition and Subtraction

36. A third way of finding the perimeter. We have seen that the perimeter may be found in two ways.

1. *Arithmetically.* Each side is measured and the measures are added. The resulting arithmetic sum is the perimeter.

2. *Algebraically.* The sides are added without finding the length of each. Thus, $p = a + b + c$. The sum $a + b + c$ denotes the perimeter. The equation $p = a + b + c$ is a formula for finding the value of the perimeter when the values of a, b, and c are given.

The following is a *third* way of finding the perimeter.

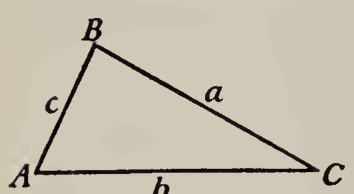


Fig. 62

Draw a line of indefinite length, as OX (Fig. 63).

Open the compass a distance a (Fig. 62) and from O lay off on OX the segment OD equal to a.

Similarly, from D lay off a distance DE equal to b in the direction of X, and from E lay off EF equal to c.

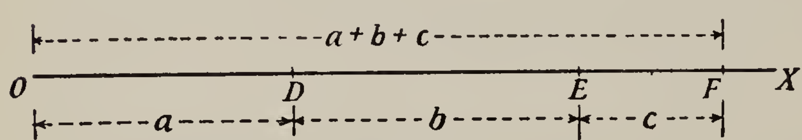


Fig. 63

OF is the perimeter of triangle ABC.

The line-segment OF may now be measured to determine the arithmetic value of $a + b + c$.

Exercises

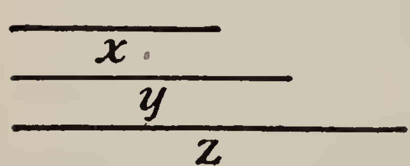


Fig. 64

1. Draw three segments, x, y, and z (Fig. 64) and find the sum in the three ways shown above.

To find the line-segment representing the sum proceed as in Fig. 63, and then mark the diagram as shown there.

2. Find in three ways the perimeter of the quadrilateral $ABCD$ (Fig. 65).

3. Draw two unequal segments a and b. Construct the sums $a+b$ and $b+a$. Show that, by measuring, $a+b=b+a$.

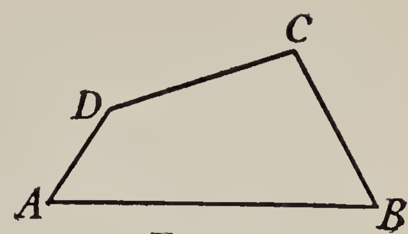


FIG. 65

37. The law of order in addition. Exercise 3 illustrates the principle that *the value of a sum remains unchanged when the order of the addends is changed.* This is called the *law of order in addition.* This law helps us to simplify the adding of numbers by arranging them first in an advantageous order. For example, the terms in the sum $475+210+25$ may be taken in the order

$$475+25+210=500+210=710$$

To indicate that we wish to add first 475 and 25 and then 210, the last statement may be written in the following forms:

$$(475+25)+210=500+210=710,$$
or $$[475+25]+210=500+210=710,$$
or $$\{475+25\}+210=500+210=710.$$

38. Parentheses. The symbols ( ), [ ], { } are used to show the *order* in which numbers are to be added, subtracted, or multiplied. Thus, $4\times(2\times3)$ means that first 2 is to be multiplied by 3 and the product is then multiplied by 4.

EXERCISES

In each of the following exercises state the meaning of the symbols and then add, subtract, and multiply in the order indicated.

1. $(324+15)+75$.

2. $(24+8)-22$.

3. $(15-2)+4$.

4. $3[5+(7+3)]$.

5. $2\{16-(7+3)\}$.

6. $18-[10-(6+2)]$.

7. Rearrange in the most advantageous way and then add: $240+325+60$; $736+298+64$; $350+287+50$.

8. State the meaning of each of the following expressions.
$(6+7)+3$; $26+(65+125)$;
$(a+b)+(c+d)$; $(10+15)+50+13$.

The symbols ( ), [ ], and { } are called **parentheses, brackets,** and **braces,** respectively.

39. Finding the difference between two segments. Draw two segments a and b (Fig. 66), making a greater than b.

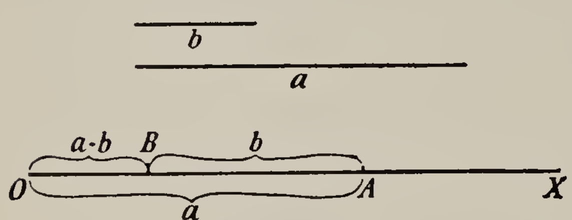


FIG. 66

Draw an indefinite line, OX.

On OX lay off $OA=a$.

From A lay off, in the direction AO, the segment $AB=b$.

The segment OB is the *difference* between a and b. In the form of an equation this may be written

$$OB=a-b$$

EXERCISE

Draw three segments a, b, c (Fig. 67) making $a>b>c$. Then construct the following sums and differences, marking all segments as shown in Figs. 63 and 66:

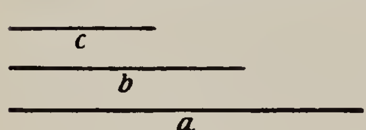


FIG. 67

$a+b-c$; $a+c+b$; $a+c-b$; $b-c+a$; $a-c+b$.

40. Monomials. Terms. Numbers like $30\times n$, $20\times t$, $r\times t$, are *monomials*, or *terms*. The numbers 30 and n are *factors* of $30\times n$, 20 and t are *factors* of $20\times t$, r and t are *factors* of $r\times t$.

41. Similar terms. Monomials having a *common* (the same) literal factor, as $4 \times a$ and $8 \times a$, are **similar** terms.

42. Coefficient. In the monomials $2 \times n$, $30 \times x$, $20 \times t$, the arithmetical factors 2, 30, 20, are called **coefficients** of the literal factors n, x, and t, respectively. It is customary to write products like $2 \times n$, $30 \times w$, $20 \times t$, briefly $2n$, $30w$, $20t$, omitting the multiplication sign. It must be remembered that such numbers have two meanings. For example, $3m$ means $m+m+m$, and 3 multiplied by m. State two meanings of each of the products above.

When no coefficient is stated, as in the numbers a, x, y, the coefficient is *understood* to be 1. Thus a means $1a$, x means $1x$.

Exercises

1. State the following products in a brief form: $6 \times s$, $5 \times m$, $80 \times d$, $p \times r$.

2. Give several examples of monomials; of similar monomials; of dissimilar monomials.

3. State two meanings for each of the following: $20y$, $10x$, $15a$. What is the meaning of ab, xy, rt?

4. Find the values of $15t$ when $t = 1$, 2, 6, 7.5, 10.75, $12\frac{2}{3}$.

43. Polynomials. Numbers like $a+b+c$ and $x+y+z$, which consist of two or more *monomials* (terms), are called **polynomials** (*many-termed*). A polynomial with only *two* terms, as $a+b$, is a **binomial**. A polynomial containing *three* terms, as $a+b+c$, is a **trinomial**.

44. Adding and subtracting similar terms. We have seen that

$$2a \text{ means } a+a,$$

and that $3a$ means $a+a+a$,

Hence, $2a+3a$ means $(a+a)+(a+a+a)$,

or $a+a+a+a+a$, or $5a$.

It follows that

$$2a+3a=5a.$$

This shows that the similar terms $2a$ and $3a$ may be *combined*, or collected, *by adding the coefficients* 2 and 3, *and then multiplying the sum by the common factor a*.

Numbers not having a common factor cannot be combined. Thus in the sum of a and b the addition must remain *indicated*, as $a+b$.

Exercises

1. By combining similar terms, reduce each of the following polynomials to a simpler form. Arrange the work as follows:

$$4m-\tfrac{1}{3}m=(4-\tfrac{1}{3})m=\tfrac{11}{3}m.$$

$$8x+4x;\ 10a+3a+a;\ \tfrac{1}{2}x+\tfrac{3}{4}x+\tfrac{1}{8}x;\ 5.2m+6.7m-3.1m.$$

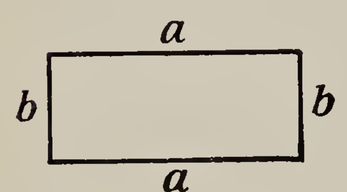


Fig. 68

2. The lengths of the sides of the rectangle (Fig. 68) are respectively a, b, a, and b. Write the perimeter in the simplest form.

3. Find the perimeter of each of the figures below (Fig. 69), write the results in the simplest form, and find the value of each of the resulting polynomials for $a=6$, $b=3$, $c=2$, $d=4$, $f=1$.

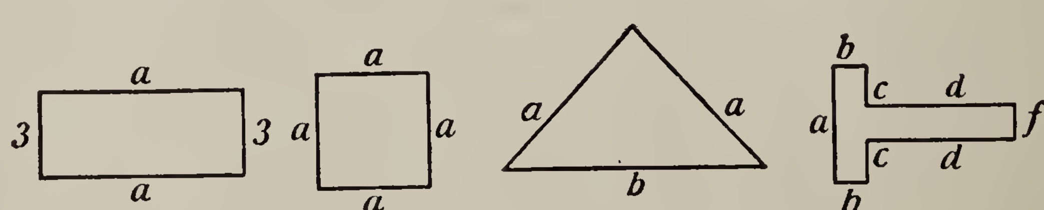


Fig. 69

4. The number of pupils in a class is n. Five pupils are withdrawn. Express as a binomial the number left.

5. A man had a acres of land and sold s acres. How many had he left?

6. The difference of two numbers is 20. The smaller number is n. What is the larger?

7. Express in symbols: $\frac{1}{6}$ of a number n diminished by 4.

8. Five times a number x is increased by 3, and the sum is divided by 7. Write the result in symbols.

Write the following polynomials in the simplest form.

9. $2x+7x-4x+6x$.
Solution: $2x+7x-4x+6x=2x+7x+6x-4x=15x-4x=11x$.

10. $3a+7a+2a-a$.

11. $\frac{1}{3}m+\frac{5}{6}m+\frac{3}{5}m-\frac{1}{2}m$.

12. $2.3t+1.8t-.9t+6.1t$.

13. $20m+6m+17n-3n$.
Solution: $(20m+6m)+(17n-3n)=26m+14n$.

14. $14x-3.6x+8.4y-1.9y$.

15. $1\frac{2}{3}a-\frac{1}{4}a+\frac{3}{5}b-\frac{1}{2}b$.

16. $12.6c-.25c+18.3c$.

17. $(8x+4x)+(10x-2x)+3$.

18. $\frac{1}{3}a+\frac{3}{8}a+\frac{1}{4}a+5$.

Find the value of each of the following for $a=6$, $b=3$, $c=2$.

19. $a+2b+3c$.
Solution: $a+2b+3c=6+(2\times 3)+(3\times 2)=6+6+6=18$.

20. $2a+\frac{b}{2}-2c+4$.

21. $\frac{a-c+2b}{3+2a}$.

22. $\frac{2c+1.2a-.6b}{a+b+c}$.

23. $\frac{3c+2b-a+1}{8c-b-a}$.

24. $\frac{\frac{1}{2}a-b+c+6}{a-\frac{2}{3}b+\frac{3}{4}c}$.

25. $\frac{6.2a-2.4b+c}{\frac{3}{4}a+1.2b+.6c}$.

EQUATIONS

45. How to solve equations. We have seen (§§30–32) that numerical facts may be represented by equations. For example, the equation $c = 30n$ states that the number of cents paid for oranges is 30 times the number of oranges. In studying perimeters (§34) we were able to find the lengths of sides of polygons by means of equations, such as $6x = 120$. In the study of mathematics the equation is a powerful instrument in solving problems and in representing numerical facts in a brief form. Hence, we must learn to work with equations.

An equation, as $6x = 30$, states the equality of the numbers $6x$ and 30. The number $6x$ to the *left* of the equality sign is called the *left side*, or the *left member*, of the equation; the number to the *right* is the *right side*, or *right member*. The literal number is called the *unknown number*. The process of finding a value of the unknown number for which both members are the same is called *solving the equation.*

EXERCISES

Solve the following equations:

1. $2x = 19.$

Solution: $2x = 19.$

Dividing each member by 2, we have

$$\frac{\cancel{2}x}{\cancel{2}} = \frac{19}{2}$$

$$\therefore x = 9.5.$$

2. $3x = 18.$

3. $2x = 11.$

4. $5x = 20.$

5. $7x = 13.$

6. $2.5\,x = 50.$

7. $3.75x = 100.$

For each of the following problems first state the equation and then solve it. Arrange your work as in the solution of exercise 8, below.

8. A certain number multiplied by 6 gives the product 42. Find the number.

Solution: Let n be the required number.

Then the problem is briefly stated in the form of the equation $6n=42$.

To find n, divide each member of the equation by 6.

This gives $\frac{\cancel{6}n}{\cancel{6}}=\frac{42}{6}$

$\therefore n=7.$

9. A train traveling at the rate of 35 miles an hour made a distance of 75 miles. How much time did it take?

10. A field containing $\frac{5}{8}$ of an acre is sold for $300. What is the price per acre?

11. In a given time the minute hand of a clock passes over 12 times as many minute spaces as the hour hand. Over how many minute spaces does the hour hand pass in 50 minutes?

12. How long will it take a man to ride 60 mi. at the rate of 18 mi. an hour?

13. A man earns $125 in 6 days. How much does he get per day?

46. Solving equations by using an axiom. In the solution of equations (§45) it has been taken for granted that when both members of an equation are divided by the same number, the quotients are also equal.

Thus, if $2x=10,$

it is assumed that $\frac{2x}{2}=\frac{10}{2},$

or that $x=5.$

This assumption may be stated in the form of a general principle as follows: *If equal numbers are divided by the same or equal numbers, the quotients are equal.*

Such statements when assumed to be true are called *axioms.*

The equation above is solved by dividing both members by 2. The principle used in the solution of the equation is called *division axiom.* The axiom is to be used in solving each of the equations below.

Exercises

Solve the following equations explaining each step, and arrange the work as shown in Exercise 1, below:

Solution:	*Authorities:*
1. $2a=16$	
$\frac{\cancel{2}a}{\cancel{2}}=\frac{16}{2}$	if equal numbers are divided by the same number the quotients are equal.
$a=8.$	by reducing fractions.

2. $11a=22.$

3. $19m=76.$

4. $3.5x=24.5$

5. $11h=17.$

6. $93n=60.$

7. $45x=120$

In the following equations find the values of the unknown numbers approximately to the nearest third figure:

8. $1.23y=532.$

9. $287x=5.89.$

10. $21.2n=62.1.$

11. $7.5k=28.2.$

12. $.57r=24.2$

13. $1.32t=226.$

14. $118s=237.$

15. $.231x=462.$

Solve the following problems by means of equations, arranging the work as shown in Exercise 16, below:

16. John is able to solve twice as many problems as James. Mary can solve three times as many as James. Together they are to solve 48 problems. How many problems does each solve?

Solution: Let x be the smallest number, *i.e.*, the number of problems James solves

Then $2x$ is the number John solves

and $3x$ is the number Mary solves

$\therefore x+2x+3x=48$ since together they solve 48 problems

$6x=48$, by combining similar terms

$\therefore x=8$, the number James solves

$2x=16$, the number John solves

$3x=24$, the number Mary solves.

Notice that the solution of the problem involves the following steps:

a. The problem is read carefully, to find what number the problem calls for. In this case there are three unknown numbers.

b. One of the unknown numbers is denoted by a letter, as "x." In this case x denotes the number of problems solved by James.

c. The other unknown numbers are now expressed in terms of "x." Thus, John and Mary solve $2x$ and $3x$, respectively.

d. Then the equation is formed, and solved.

These suggestions, if followed, will be helpful in solving the problems below:

17. A sum of $32 is to be broken up into two parts, one part being 7 times as large as the other. Find the two parts.

18. Sixty rods of fence are available to inclose a rectangular field. The field is to be 5 times as long as it is wide. Make a sketch of the field and find the dimensions.

19. A man is three times as old as his son. The sum of their ages is 48 years. Find the age of each.

20. A class is to be formed having twice as many boys as girls. If there are to be 30 pupils in the class, find the number of girls.

21. The sum of three numbers is 80. The second is twice as large as the first and the third is five times as large as the first. Find the three numbers.

Solve the following equations:

22. $x+4x=200$.

23. $5x-x=60$.

24. $7x+x=30$.

25. $8x-x=35$.

26. $5x+3x=40$.

27. $5x+4x-x=70$.

47. What every pupil should know and be able to do. In the unit just completed the meaning of formula and equation has been developed. The pupil is now expected to know the following facts stated and to be able to do what the problems below call for:

1. To use correctly the expressions: polynomial, monomial, binomial, trinomial term, similar terms, value of a literal number, coefficient, formula, equation, polygon, triangle, quadrilateral, pentagon, hexagon, vertex, side, perimeter.

2. To understand the arithmetic, geometric, and algebraic forms of representing numerical facts, *i.e.*, the representation by table, graph, and formula.

3. To know the meaning of the symbols ( ), [ ], { }.

4. To find the value of expressions of the form $6x$; $a+b+c$; $2x+3x+x$; $\frac{a+b+c}{4-2a}$; where the values of the literal numbers are whole numbers, common fractions, or decimal fractions.

5. To solve equations of the form $5x=12$; $2x+7x=4$.

6. To solve simple problems leading to equations of the forms given in 5.

7. To know the following laws:

a. *Law of order in addition.*

b. *Division axiom.*

48. Typical problems and exercises. Pupils should be able to work problems and exercises of the type given below:

1. The cost of oranges of a certain size is 50 cents a dozen. Find the price of 2, 3, 4, ..., 10 dozen.

Represent these facts in tabular form; in graphical form; in a formula.

Show how to obtain the facts in the table from the formula.

From the graph determine the price of $4\frac{1}{2}$ dz., $6\frac{1}{2}$dz., 8 dozen.

2. Translate into words the statement $i=12f$.

3. The perimeter of an equilateral pentagon is 35 inches. By means of an equation find the length of each side.

4. John solved twice as many problems as Mary, and James solved three times as many as Mary. Together they solved 36 problems. How many did each solve?

5. Draw a triangle and find the perimeter by measuring each side and then adding the results; by constructing the sum and then measuring the sum.

Find the value of each of the following, if $a=6$, $b=\frac{1}{2}$, $c=1.2$.

6. $\frac{1}{3}a+\frac{3}{8}a+\frac{1}{4}a$.

7. $(8a+4b)+(6a-2c)$.

8. $8.5c+1.16a—3b$.

9. $\dfrac{3.2a-4.1b+2c}{\frac{1}{2}a-3b}$.

Solve the following equations and check the answers:

10. $45x = 120$.

11. $2.12n = 62.1$.

12. $5x + 4x - x = 70$.

13. State the following laws:
The division axiom.
The law of order in addition.

14. Write a paper on one of the following topics:
a. The value of the algebraic formula.
b. The equation as a tool for solving problems.

CHAPTER IV

A STUDY OF ANGLES

How Angles Are Used

49. Meaning of angle. After studying the line-segment, which is the simplest geometric figure, we are now able to extend our study of geometry to more complicated figures made up of several line-segments. A figure formed by two line-segments (Fig. 70) is called an *angle*. The word comes from the Latin *angulus*, meaning corner.

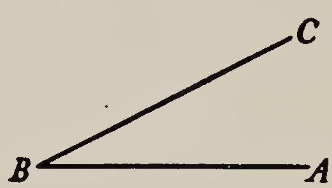


Fig. 70

More precisely, we may say that an angle is a figure formed by two straight lines, as BA and BC, passing from the same point (Fig. 70). The straight lines are the *sides*, arms, or legs, of the angle; the point, B, where the sides meet, is the *vertex*.

Exercises

1. Angles are found all about us. Point out some angles in the class room, and the sides and vertex of each.
2. Point out several angles outside of the class room.
3. Draw an angle on the blackboard.
4. Make an angle by folding a sheet of paper.
5. Draw a polygon and point out the angles in it.

50. Uses of angles. A knowledge of angles is important. Builders and architects use angles in plan-

ning and constructing our homes; the surveyor to find unknown distances which he cannot measure directly, such as the height of a tree (Fig. 71), the height of a tower or mountain, or the width of a river (Fig. 72); the astronomer to determine the position of the stars and planets, and to determine the exact time. A knowledge of angles enables the navigator guiding his ship along the coast, to avoid hidden and dangerous rocks (Fig. 103). Engineers, designers, artists, and many others, make use of angles in doing their work.

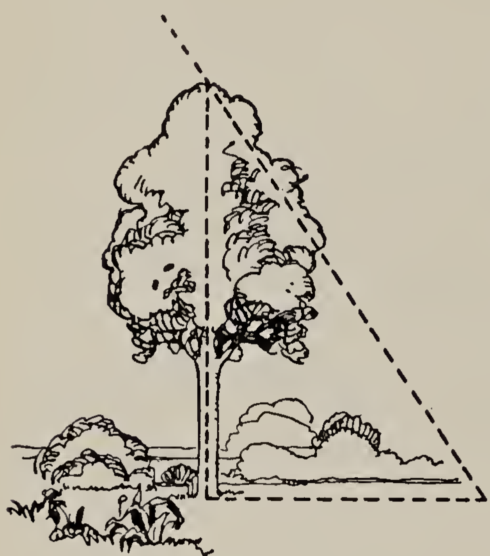


Fig. 71

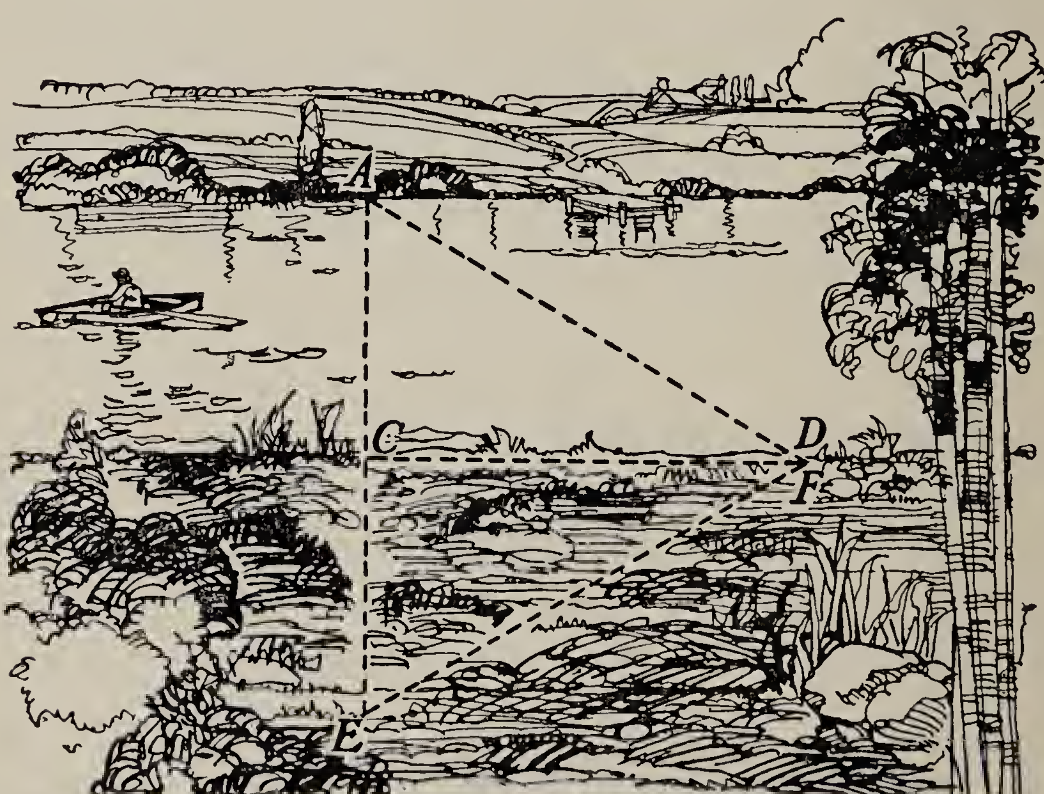


Fig. 72

By carefully studying this chapter we shall gain a thorough understanding of angles. We shall measure and draw angles to learn about their size. We shall learn the names of various kinds of angles. We shall study the relations between angles in the same geometric figure, as in the triangle or in several lines intersecting in the same point (Fig. 73), and use these relations to solve some problems in angles.

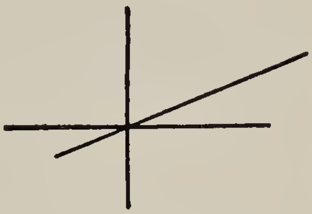


FIG. 73

We have said that an angle is a figure formed by two lines. It should not be inferred that two lines always form an angle. Point out lines in the class room which do not form an angle, no matter how far the lines may be extended. Such lines are known to us as *parallel lines*. The last part of this chapter takes up the study of angles formed by two parallel lines intersected by a third line (Fig. 74).

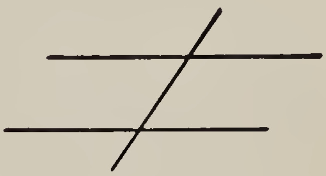
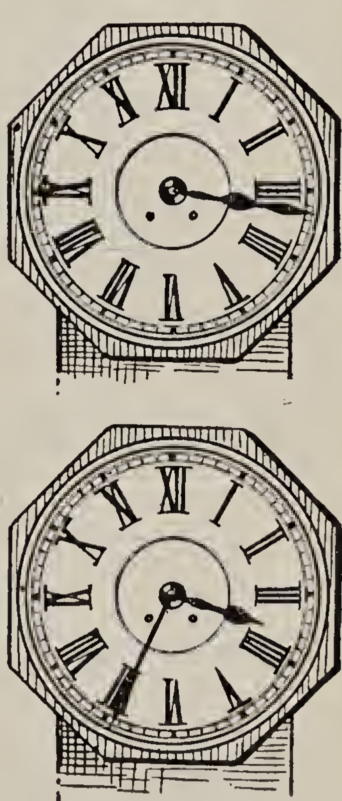


FIG. 74

51. Size of an angle. In the clock (Fig. 75) the hands are shown to be together. Since the minute hand of a clock turns faster than the hour hand, a turn of the minute hand separates the two hands, as in Fig. 76. The hands then *form an angle*. When the hands of a clock move, the angle changes in size.



FIGS. 75, 76

An angle may be formed by keeping point B (Fig. 77) fixed and then

turning a line from the position BC in the *clockwise* direction until it takes the position BA; or by turning a line from BA in the *counterclockwise* direction until it takes the position BC. *The size of the angle depends entirely on the amount of turning necessary to carry the moving line from one side to the other, and not on the length of the sides.*

Fig. 77

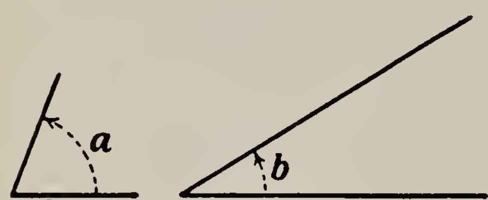


Fig. 78

Which of the two angles (Fig. 78) is the larger? Give a reason for your answer.

The curved arrow shown in the diagram is used to indicate the *direction* and *amount* of turning.

Two angles are *equal* if the same amount of rotation is needed to form them. If two angles are equal they can be made to *coincide* (fit).

Cut two angles from paper, and test them as to equality by placing one on the other. Tell which of the two is the larger angle.

52. Symbols used to denote angles. If we are to discuss angles, we must have a way of naming them. The symbol for the word *angle* is $\angle$. For *angles* it is $\angle$s.

There are various ways of naming an angle. We may use three letters (Fig. 79) one on each side and one at the vertex, and refer to the angle as $\angle ABC$, *using the vertex letter as the middle letter.*

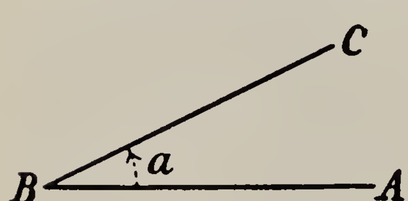


Fig. 79

A briefer notation is $\angle B$, which really means "an angle whose vertex is B."

Sometimes a small letter, as a, is written within the angle, and the angle is then called a.

Exercises

1. How many angles do you see in the triangle (Fig. 80)? Make a drawing of the triangle, and name each of the angles in the three ways described in §52. Thus, $\angle A$, a, $\angle ABC$, etc.

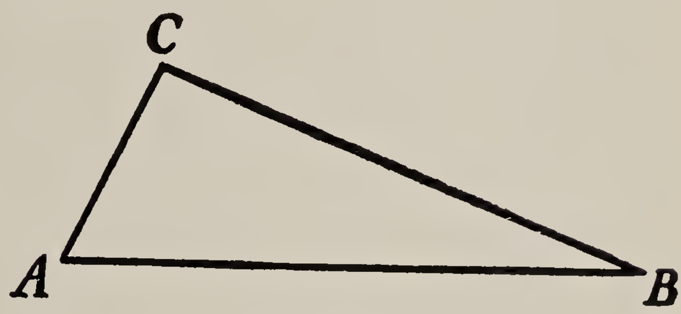


Fig. 80

2. How many angles are there in Fig. 81? Name each angle in three ways.

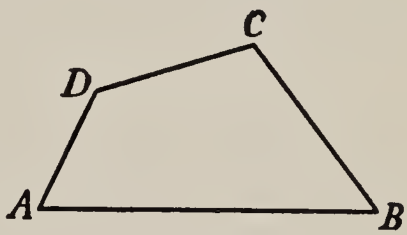


Fig. 81

3. Draw three lines passing from the same point (Fig. 82). Name the three angles marked in the figure.

4. Draw a figure like Fig. 83 and name the angles formed.

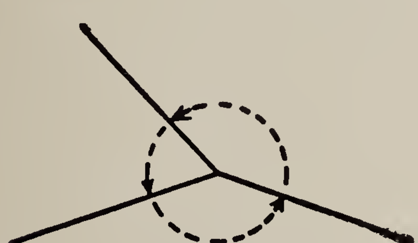


Fig. 82

Fig. 83

53. How angles are classified. Draw two lines in the position AB and AC (Figs. 84–88). Let a moving line turn about point A in the counterclockwise direction from the side AB to AC.

When the amount of rotation is equal to a *quarter turn* (Fig. 84) $\angle BAC$ is called a *right angle.*

An angle *less* than a right angle is *acute* (sharp) (Fig. 85).

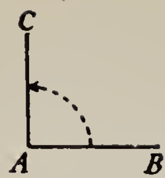


FIG. 84

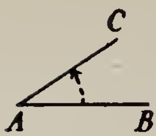


FIG. 85

FIG. 86

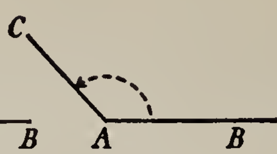


FIG. 87

FIG. 88

When AC makes a *half* turn (Fig. 86) a *straight* angle is formed. Thus, *the sides of a straight angle are in the same straight line on opposite sides of the vertex.*

Angles less than a straight angle and greater than a right angle are *obtuse* (blunt) (Fig. 87).

If the line AC makes a *complete* turn (Fig. 88) the angle is a *perigon* (round angle).

EXERCISES

1. State a time of the day when the hands of the clock form a right angle; a straight angle.

2. Point out obtuse angles in the class room.

3. Open the arms of a blackboard compass so as to form an acute angle; a right angle; a straight angle.

4. Make a sketch of each of the following: right angle; acute angle; straight angle; obtuse angle; perigon.

FIG. 89

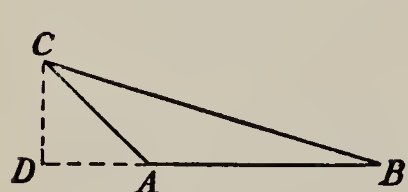


FIG. 90

5. Classify the angles a and b (Fig. 89).

6. Open your notebook making two edges form a right angle; an obtuse angle; a straight angle.

7. Name an acute angle (Fig. 90); a straight angle; an obtuse angle; a right angle; a perigon.

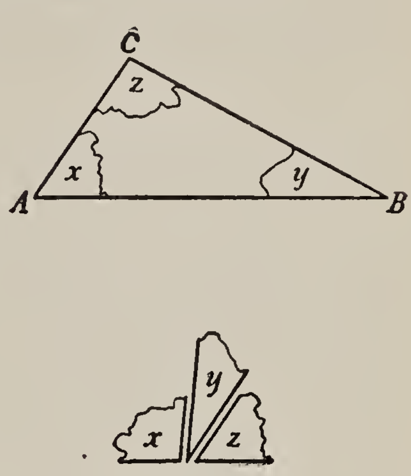
8. Classify the angles A, B, and C of triangle ABC (Fig. 90) and $\angle D$.

9. Name the largest angle (Fig. 90); the smallest angle.

10. Write in symbols: angle B is less than angle D; angle BAC is greater than angle B; angle DCA is equal to angle CAD.

11. How many right angles are there in a straight angle? In a perigon? How many straight angles are there in a perigon?

12. Draw a triangle ABC (Fig. 91) making the base about 8 cm. long. Mark the angles x, y, and z, respectively. Cut the triangle from the paper and tear off the corners as shown in the diagram. Place them adjacent to each other as in Fig. 92. Show that *the sum of the three angles of a triangle is a straight angle.* This is an important fact of geometry.



FIGS. 91, 92

13. Draw a quadrilateral. Tear off the corners, and show as in Exercise 12 that the sum of the four angles is a perigon.

MEASURING ANGLES WITH THE PROTRACTOR

54. Protractor. The *protractor* (Fig. 93) is an instrument used mainly for *measuring* angles. The curved rim is divided into 180 equal parts, every tenth of which is numbered. A line drawn from O, the mid-point of the straight edge, through a mark B on the rim, as OB, forms with the zero-lines, OX and OY, angles whose sizes may be read off on the outer and inner readings, respectively. Thus the measure of the straight angle XOY is 180, the measure of angle XOB is 30.

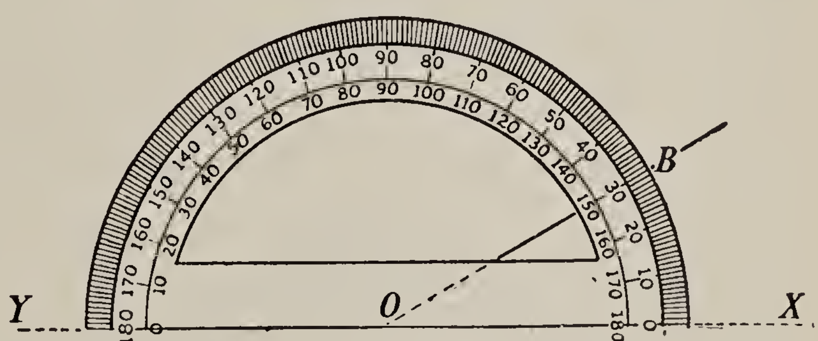


FIG. 93

If from the point O, lines are drawn to the marks on the rim, they divide the straight angle into 180 equal small angles, each of which is a *degree.*

The division of a straight angle into 180 equal parts, is brought down to us from the Babylonian astronomers. For scientific purposes it would have been more convenient to use the metric system of dividing into 10, 100, 1000, etc., equal parts. Accordingly a right angle would be divided into 100 rather than into 90 equal angles. Only very recently attempts have been made in France to establish such a system.

The division of a round angle into 360 equal parts, or of a straight angle into 180 equal parts, originated as follows. The Babylonian astronomers believed that the sun revolved around the earth in 360 days. The daily part of a revolution was represented by the 360th part in a perigon. A knowledge of the triangle having equal sides probably led them to use one of the 3 equal angles as a unit. This unit being contained 6 times in the perigon was accordingly sub-divided into 60 equal angles. This led to further division of these angles into 60 equal parts. This is known as the *sexagesimal system.* In this system every unit is divided into 60 equal parts. It is interesting to note that even today, modern civilization is still influenced by ancient Babylonian science in the reckoning of time, and angles. For, we divide an hour into 60 minutes, and a minute into 60 seconds. Similarly, the circle is divided into 360 equal parts called arc degrees. A degree of the arc is divided into sixty minutes, and a minute of arc into sixty seconds.

55. Units of angular measurement. We have seen that the 360th part of a *perigon* is a *degree;* that a degree is also the 180th part of a straight angle, and the 90th part of a right angle.

A *degree* may be divided into 60 equal parts. They are called *minutes* (1′).

A *minute* may be divided into 60 equal parts, called *seconds* (1″). These facts are summarized in the *table of angular measure* below. Since this table is used in future problems it should be memorized.

TABLE OF ANGULAR MEASURE

1 perigon	$=360°$
1 st. ∠	$=180°$
1 rt. ∠	$=90°$
1°	$=60'$
1′	$=60''$

EXERCISES

1. Reduce 20° 15′ 18″ to seconds.

Solution:
$$\begin{aligned} 20° &= 1200' = 72000'' \\ 15' &= 900'' \\ 18'' &= 18'' \end{aligned}$$

Adding, $20°+15'+18''=72918''$

2. Reduce 17° 59′ 20″ to seconds.

3. Reduce 435′ to degrees and minutes.

4. Reduce 1026″ to degrees, minutes, and seconds.

5. Measure ∠*XOA* (Fig. 94).

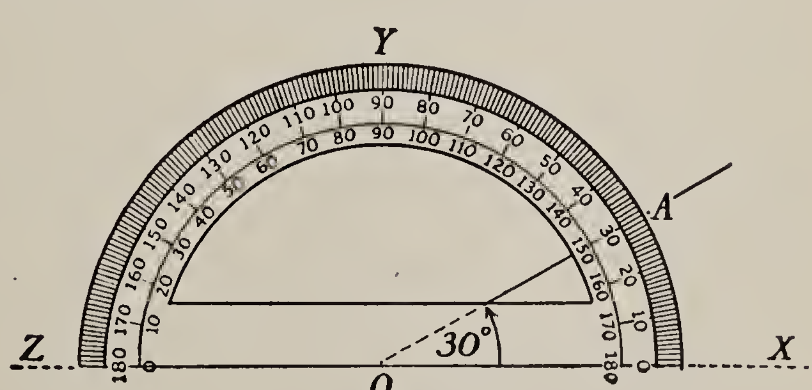


FIG. 94

Explanation: Place the protractor with the center on the

vertex O and make the line from the center to the zero-mark on the right of O fall along the side OX.

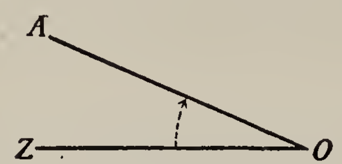


Fig. 95

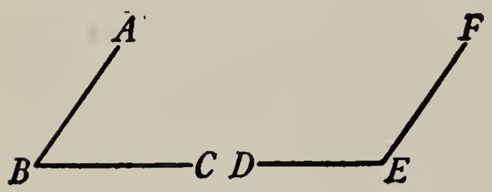


Fig. 96

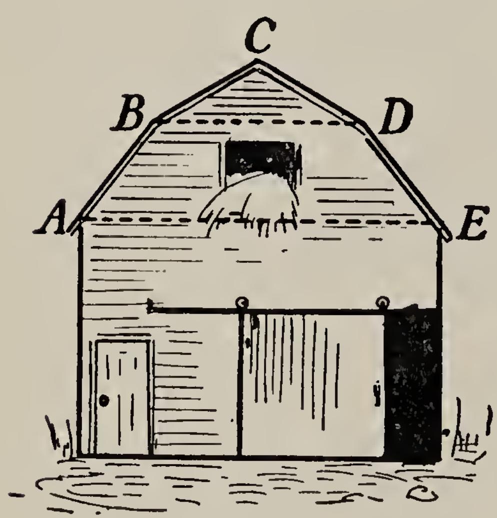


Fig. 97

From the zero-mark pass along the rim to the point where OA, or the extension of OA, intersects it.

Then take the reading at this point. This is the measure of $\angle XOA$.

Hence, $\angle XOA = 30°$.

6. Measure $\angle ZOA$ (Fig. 95).

Suggestion: Place the zero-line of the protractor along OZ. From the left-hand zero-mark pass along the rim to the point where OA, or the extension of OA, intersects it. Write the result near the vertex inside of the angle. To check your result measure the angle again by placing the zero-line along OA.

7. Measure ∡ ABC and DEF (Fig. 96). Check your results, as in Exercise 6.

8. Fig. 97 is a picture of one end of a barn. Find by measuring what angles the rafters AB and BC form with each other. Find at what angle the rafter AB inclines toward the horizontal beam AE.

9. Draw two angles. Measure each, and find the ratio to two figures.

10. Draw a triangle whose longest side is 8 centimeters. As in §53 classify each angle of the triangle, and record the results in the table on page 85.

Estimate the number of degrees in each angle. Record the results in the table.

Measure each angle, and record the results.

By taking the difference between degrees estimated and degrees measured, compute errors, and record them in the column headed *errors*.

Find the sum of the number of degrees obtained by measuring.

Angles	*Classification of Angles (Acute, Obtuse, Etc.)*	*Number of Degrees*		*Errors*
		Estimated	*Measured*	
A				
B				
C				
Sums				

Compare your sum with the results of the other pupils of the class. What seems to be the number of degrees in the sum of the angles of a triangle?

56. Sum of the angles of a triangle. In Exercise 12 (§53) we have seen that the sum of the angles of a triangle is a straight angle. In §55, we have found by measurement that the sum of the angles of a triangle is 180 degrees. This may be expressed in symbols by means of the equation, $a+b+c=180$.

This equation is used by the surveyor as a formula for finding one angle of a triangle if the other two are known.

Exercises

Solve the following exercises using the formula $a+b+c=180$.

1. Make a triangle so that the first angle is three times the second, and the third is six times the second.

Solution: Let x be the number of degrees in the second angle.

Then $3x$ is the number of degrees in the first, and $6x$ is the number of degrees in the third.

$x+3x+6x=180$, for the sum of the angles of a triangle is 180°.

Hence, $10x=180$, by combining similar terms.

$x=18$, by dividing both sides of the equation by 10.

$\therefore 3x=54$,

and $6x=108$.

Check: $x+3x+6x=180$.

2. The three angles of a given triangle are equal. Find them.

Suggestion: Let x be the number of degrees in each angle, form an equation containing x, and solve.

3. The first angle of a triangle is twice as large as the second, and the third is 3 times the first. Find the value of each.

Suggestion: Let x be the number of degrees in the second angle.

4. Two angles of a triangle are equal and the third is equal to the sum of the other two. Find each angle.

5. The first angle of a triangle is four times as large as the second, and the third is one-half the first. Find each angle.

6. Find the angles of a triangle if the first is 6 times as large as the second, and the third one-half as large as the first.

7. The first angle of a triangle is one-half as large as the second, and the third is three-fourths as large as the second. Find each angle.

Solution: Let n be the number of degrees in the second angle.

Then $\frac{n}{2}$ is the number of degrees in the first,

and $\frac{3n}{4}$ is the number of degrees in the third.

Then $n+\frac{n}{2}+\frac{3n}{4}=180$,

$$\therefore \frac{9n}{4}=180,$$

$$\therefore n=80,$$

$$\frac{n}{2}=40,$$

$$\frac{3n}{4}=60.$$

8. The first angle of a triangle is $\frac{1}{3}$ of the second, and the third is $\frac{1}{2}$ of the first. Find the three angles.

57. Measurement of angles in surveying. A simple instrument for measuring angles out of doors may be made by a pupil as follows:

On a drawing board (Fig. 98) fasten a large protractor. By means of a pin stuck into the board at O

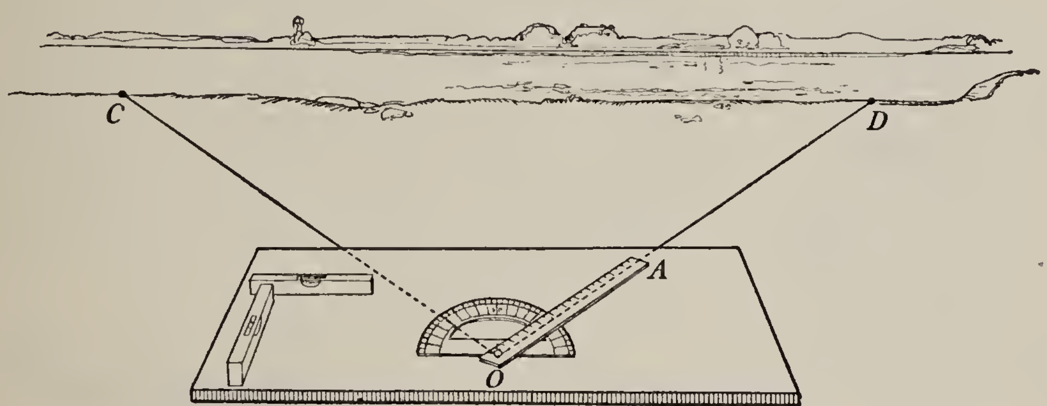


FIG. 98

attach a ruler so that it may move freely over the board when it is made to turn about O. A pin stuck in at A near the other end of the ruler may be used for sighting.

The board may be placed on a tripod, or a table, and brought into horizontal position by means of two spirit levels attached as shown in the figure.

To measure $\angle DOC$, sight in the direction of the side OD and take the reading on the protractor. Repeat this for the side OC. The difference between the two readings is the *measure* of the angle.

Surveyors and astronomers measure angles with a transit (Fig. 99). The telescope on this instrument is used for *sighting* along the sides of the angle which is to be measured. The *measure* of the angle is found by

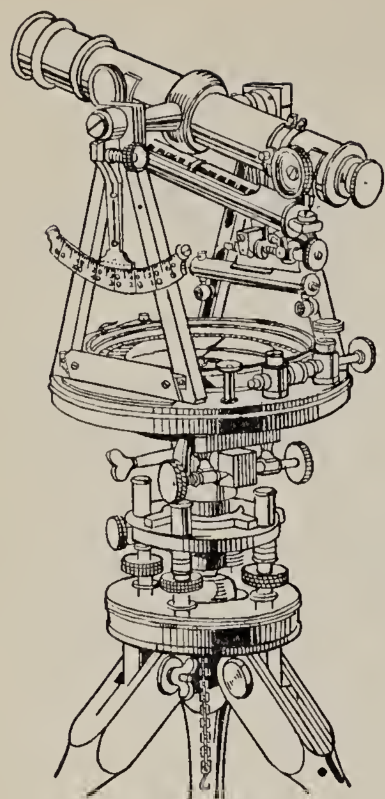


Fig. 99
Surveyor's Transit

means of two graduated circles which serve as protractors. One of these is used to measure *vertical* angles, as AOB (Fig. 100) and the other for measuring *horizontal* angles, as COD (Fig. 98).

58. Measurement of angles in navigation. By measuring angles the navigator is able to avoid dangerous obstacles, as rocks or sand-banks (Fig. 103), which lie near the course of his ship. He uses for this

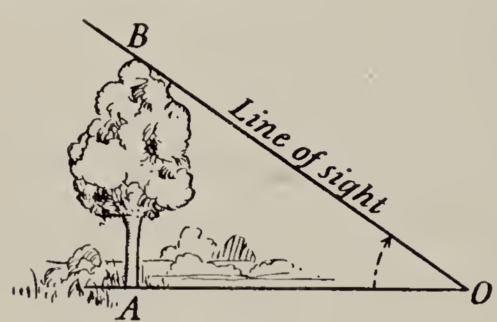


Fig. 100

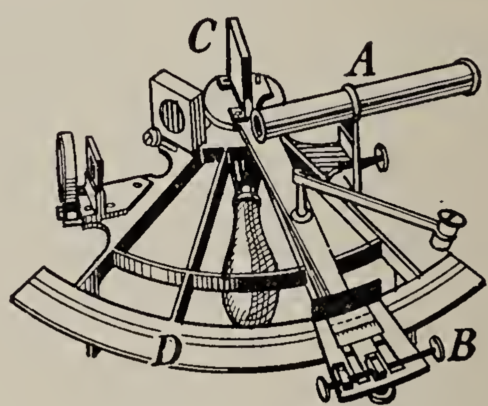


Fig. 101. Sextant

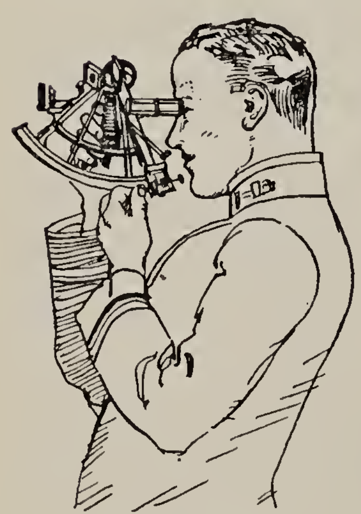


Fig. 102
Measuring an Angle with a Sextant

an instrument called the sextant (Fig. 101).

The instrument is used to measure an angle whose vertex is the observer's eye and whose sides pass through 2 remote objects. It is held by the handle in the right hand, with the telescope A (Fig. 101) toward the observer's eye (Fig. 102). Making the line of sight pass

through one of the objects the sliding arm B is moved until the image of the second object is reflected by the mirror C into the telescope. The angle is found by means of the scale on the arc D of the sextant.

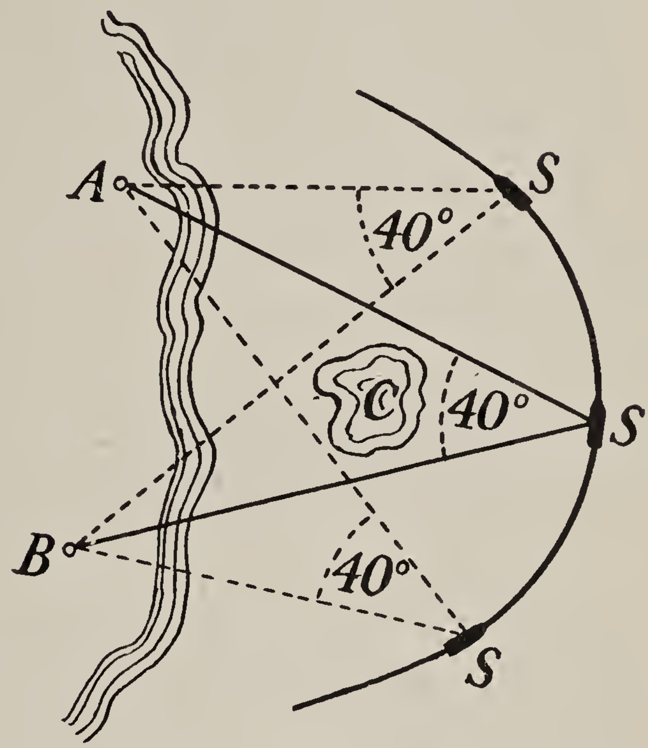


FIG. 103

As the ship S (Fig. 103) approaches the dangerous region C, two well defined objects A and B on the shore line are constantly observed from the ship with the sextant. This gives repeated measures of angle ASB. The ship must round the obstacle in such a way that angle ASB does not increase in size.

FIG. 104. MARINER'S COMPASS.

The *mariner's compass* (Fig. 104) is used to show the *direction* (angle between the course of

the ship and the north-south line) in which a ship is traveling.

The rim of the compass is divided into 32 equal parts, called *points*, each of which has a name as shown in the picture.

State the size of the angles formed by the following directions: *E* and *SE*, *NE* and *W*, *N* and *E*, *SE* and *NW*, *WSW* and *E*, *S* and *NW*.

59. Adjacent angles. Two angles ABC and DEF (Fig. 105) may be placed so that the vertices *E* and *B* coincide (Fig. 106) and that side *ED* falls along side *BC*. In this position the angles are said to be *adjacent* to each other. In general, two angles are **adjacent** if they have a *common* (the same) vertex and a common side between them.

The other two sides, *BA* and *BF*, are called *exterior* sides.

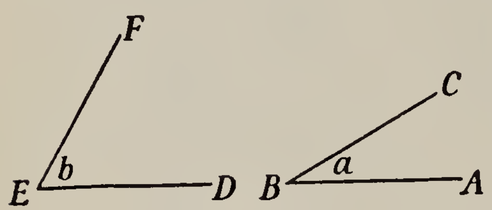


Fig. 105

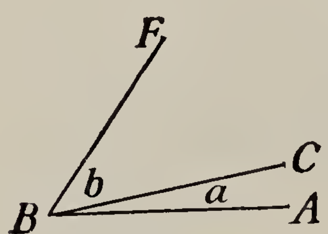


Fig. 106

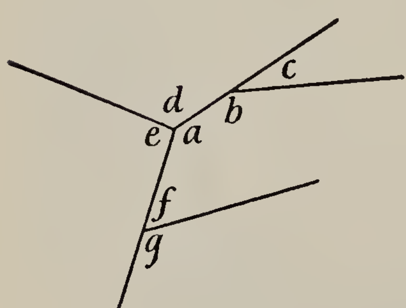


Fig. 107

Fig. 108

Exercises

1. State whether or not the following angle pairs (Fig. 107) are adjacent and give reasons for your answers: *a* and *b*, *b* and *c*, *c* and *d*, *d* and *e*, *e* and *f*, *f* and *g*.

2. Make sketches showing two acute adjacent angles; two obtuse adjacent angles.

3. Find adjacent angles in the class room.

4. Measure the adjacent angles m and n (Fig. 108).

5. Draw two intersecting lines making one of the adjacent angles 4 times as large as the other. Begin by drawing a sketch. Then find the angles by means of an equation.

Solution: Let x be the number of degrees in one angle.
Then $4x$ is the number of degrees in the adjacent angle,
and $4x+x=180$.

Solve this equation.

Make an *accurate* drawing.

6. One of two adjacent angles formed by two intersecting lines is 3.5 times as large as the other. How large is each angle?

7. One of two adjacent angles is $\frac{1}{5}$ as large as the other. Find the number of degrees in each.

60. Perpendicular lines. Two straight lines forming *equal adjacent* angles are **perpendicular** to each other. Thus the equation $m=n$ (Fig. 109) states in symbols that HF is perpendicular to EG. Another symbol for perpendicularity is $\perp$. Hence, the statement *HF is perpendicular to EG* may be written briefly $HF \perp EG$.

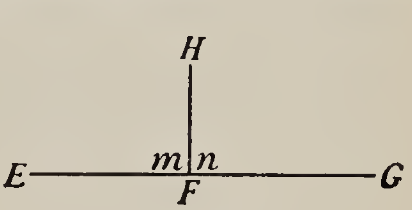


Fig. 109

Exercises

1. State in two ways that the lines in Fig. 110 are perpendicular to each other.

2. Two lines may be perpendicular to each other without intersecting. Which of the lines in Fig. 111 are perpendicular? Give reasons.

Fig. 110

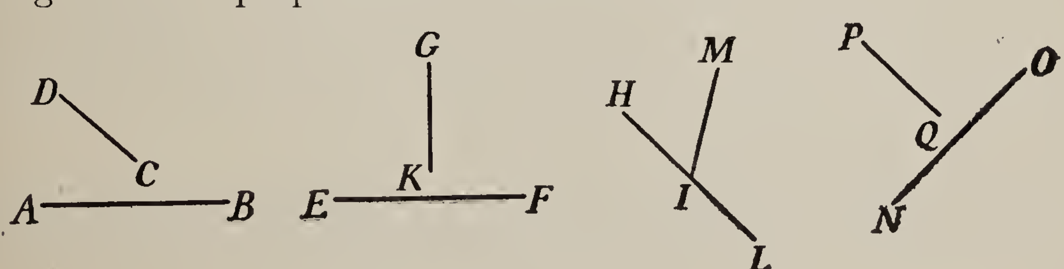


Fig. 111

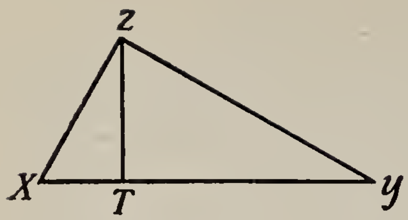


FIG. 112

3. Locate perpendicular lines in the class room; in Fig. 112.

4. Prove by means of an equation that if two lines are perpendicular to each other (§60) the two adjacent angles are right angles (Fig. 113).

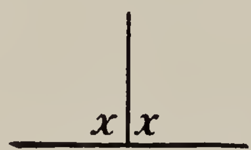


FIG. 113

5. Exercise 4 suggests the following use of a triangle having one right angle for drawing perpendicular lines.

Draw a line perpendicular to AB (Fig. 114) and passing through C.

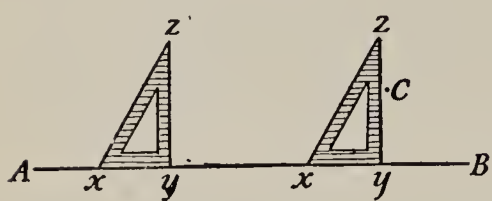


FIG. 114

Directions: Place a right triangle XYZ so that one side of the right angle lies on AB. Slide the triangle along AB until YZ passes through C. Then draw a line along YZ. This is the required line.

61. Supplementary angles. Measure angles m and n (Fig. 115) and find the sum. Two angles whose sum is 180°, or a straight angle, are **supplementary angles.** Each is said to be the *supplement* of the other.

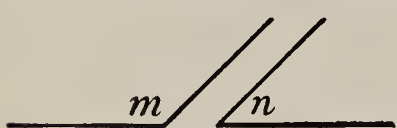


FIG. 115

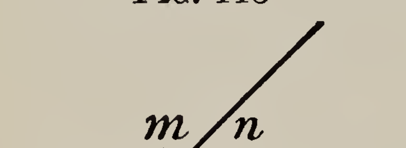


FIG. 116

When two supplementary angles are *adjacent* (Fig. 116) they are called *adjacent supplementary angles.*

EXERCISES

1. Using only a ruler, draw a sketch of two supplementary angles. Measure with a protractor each of the angles and test the accuracy of your drawing.

2. How many degrees are there in the supplement of 45°? of 20°? of 160°? of 130°? Tell how the supplement is found in each case.

3. Find the supplement of $67° 15'$.

Solution: $180° = 179° 60'$,
Subtracting $67° 15'$,
we have the supplement $= 112° 45'$.

4. Find the supplement of $110° 30'$; of $25° 40'$; of $18° 57'$; of $90° 40' 32''$, arranging your work as in exercise 3.

5. Find the supplement of $a°$; of $x°$. Write the result in the form of binomials.

6. Make a formula for finding the supplement, s, of any given angle, a.

7. State by an equation that $a°$ and $b°$ are supplementary.

8. State by equations that the following pairs of angles are supplementary

$x°$ and $50°$; $x°$ and $\frac{1}{2}x°$; $(x+20)°$ and $(2x-4)°$.

9. One of two supplementary angles is 5 times as large as the other. Find the two angles by means of an equation.

10. One of two supplementary angles is .8 as large as the other. Find the two angles.

11. Find two supplementary angles, if one is $1\frac{4}{7}$ times as large as the other.

62. Complementary angles. Measure angles x and y (Fig. 117) and find the sum.

Two angles whose sum is $90°$, or a right angle, are **complementary angles.** Each is the **complement** of the other.

FIG. 117

EXERCISES

1. Make a sketch of two complementary angles. Test your drawing with a protractor.

2. Make a sketch of two adjacent complementary angles.

3. Arranging your work as in Exercise 3 (§61) find the complement of 30°; 70°; 80.5°; 27° 14′; 18° 25′; 16° 13′ 40″; 65° 25′ 32″.

4. Find the complement of $a°$; $x°$. Write the results as binomials.

5. Make a formula for finding the complement, c, of a given angle a.

6. State by means of an equation that 20° is the complement of $a°$; that $x°$ is the complement of $\frac{5x°}{4}$.

7. One of two complementary angles is 8 times as large as the other. Find the angles by using an equation.

8. A right angle is to be divided into two parts so that one is $5\frac{1}{2}$ times as large as the other. Find the two parts.

9. Draw a right triangle. Show by measuring that the acute angles are complementary. Naming the acute angles a and b, state the equation.

10. Find the acute angles of a right triangle if one is three times the other; $\frac{2}{3}$ as large as the other. Make sketches of the angles.

63. Opposite angles. Draw two intersecting lines, as AB and CD (Fig. 118). Measure the angles m, n, r, and s. Into each angle write the number of degrees it contains. How do angles m and r compare as to size? Compare angles n and s.

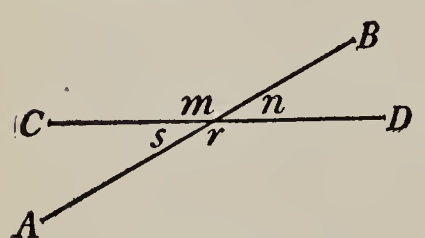


Fig. 118

State by means of equations the relations between m and n; m and s; s and r; r and n. What are these angle pairs called?

State the relations between m and r; s and n.

The angle-pairs m and r, n and s are called *opposite*,

or vertical, angles. **Opposite angles** are formed by two intersecting straight lines so that the sides of one angle lie in the same straight lines as the sides of the other, but in opposite directions from the vertex.

From the measures of the angles (Fig. 118) it is seen that *if two lines intersect, the opposite angles are equal.*

Exercises

1. Draw two intersecting straight lines and name the opposite angles. Express by equations the fact that the opposite angles are equal.

2. Two intersecting lines make one angle equal to 32°. Find the other angles.

3. Find the sum of the angles just covering the plane around a point (Fig. 119). Express the result in the form of an equation.

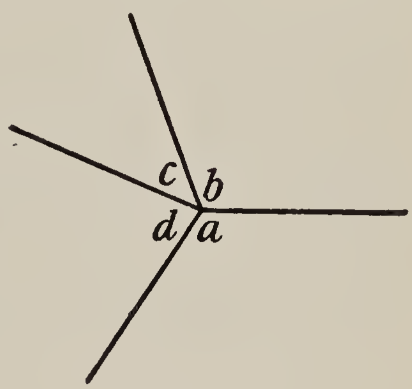


Fig. 119

64. Summary of angle relations. Express the relations for the following angles in the form of equations; in words; and by means of figures.

1. The angles of a triangle.
2. Two supplementary angles.
3. Two complementary angles.
4. Two opposite angles.
5. The acute angles of a right triangle.
6. The adjacent angles formed by two perpendicular lines.

Drawing Angles with the Protractor

65. To draw an angle of a given size. To draw an angle of 45°, draw first a straight line, as ABC (Fig. 120). Then place the center of the protractor at B and the zero-mark exactly on BC.

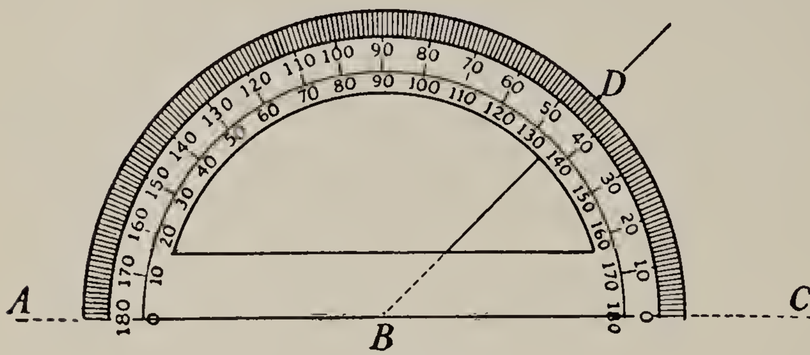


Fig. 120

Starting at the zero-mark, pass along the rim and place a point D at the 45° mark.

Remove the protractor and draw a line from B passing through D.

Angle CBD is the required angle.

Exercises

1. Using the protractor and straight edge, draw an angle equal to 30°; 90°; 120°; 180°; $65\frac{1}{2}$°; $94\frac{1}{2}$°.

2. By means of the protractor draw a line perpendicular to a given line BC, at one of its points, as A.

3. Draw a triangle having a right angle.

66. To draw an angle equal to a given angle. In making designs like those of Fig. 121 one must be able to draw angles of the same size as the angles in the required design.

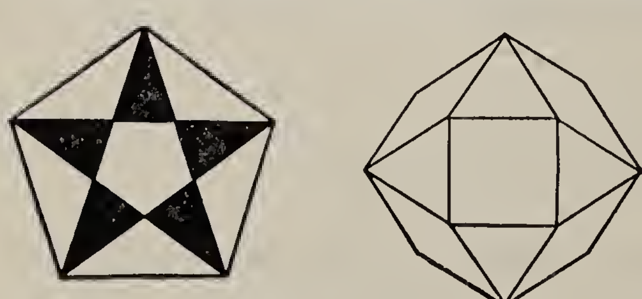


Fig. 121

This may be done by means of the protractor. For example, let it be required to draw an angle equal to $\angle ABC$ (Fig. 122). Measure $\angle ABC$ and write the number of degrees it contains inside of the angle, near the vertex.

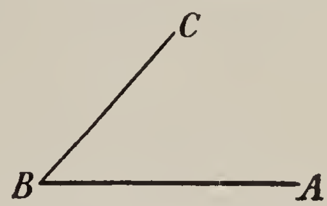


FIG. 122

Draw a line as DEF (Fig. 123), and by the method of §65, draw on DF at the point E an angle containing the same number of degrees as ABC.

FIG. 123

This is the required angle.

EXERCISES

1. With a protractor and ruler draw a triangle having one right angle. What is the relation between the acute angles? Express the result in the form of an equation.

2. Using the protractor to draw the right angles, draw a square whose sides are 3 cm. long.

3. Draw a rectangle having two consecutive sides equal to 2″ and 4″, respectively.

4. Draw two intersecting lines, as AB and CD (Fig. 124). Mark a point E on CD. At E draw an angle x' equal to x.

5. Make the designs shown in Fig. 121.

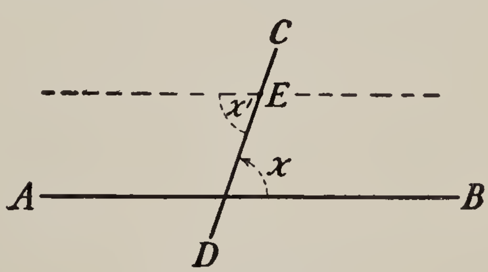


FIG. 124

6. Draw a triangle having two equal angles. To test the accuracy of your drawing measure the two sides opposite the equal angles. Compare them as to length.

This exercise shows that *if two angles of a triangle are equal the sides opposite them are equal.*

7. In the making of trusses for bridges, the beams are put together in the form of triangles, having two equal sides. Make the

design (Fig. 125). Lay off $AB = BC = CD$. Then draw angles equal to 65 degrees at A, B, C, and D.

If your drawing is well made, the triangles AEB, BFC, and CGD have two equal sides, and the points E, F, G, lie on the same straight line.

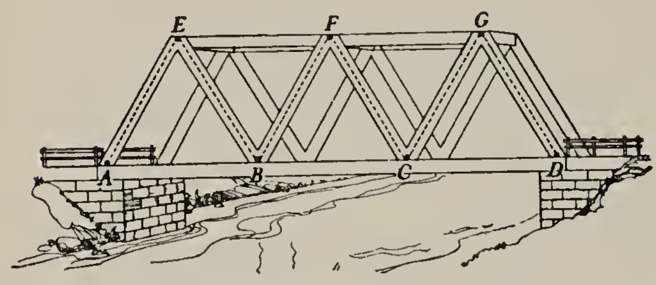


Fig. 125

8. In a triangle two angles are equal. The side opposite one of them is 6 inches long. How long is the side opposite the other?

9. Draw a triangle having each of two angles equal to 60 degrees. What must be the size of the third angle? Measure the third angle and thus test the accuracy of your drawing. Measure the sides of the triangle.

This exercise shows that *if the three angles of a triangle are equal, the three sides of the triangle are equal.*

Show that the triangles in Fig. 125 have three equal sides, if each of the angles at A, B, C, and D is 60 degrees.

10. Draw a triangle having one angle equal to 60 degrees, and another equal to 30 degrees. What is the size of the third angle? Measure the third angle to test the accuracy of your drawing.

Measure to two decimal places the side opposite the 90-degree angle, and the side opposite the 30-degree angle. Find the ratio of the two sides.

The exercise shows that *in a right triangle whose acute angles are 30 degrees and 60 degrees the side opposite the 90-degree angle (hypotenuse) is twice as long as the side opposite the 30-degree angle.*

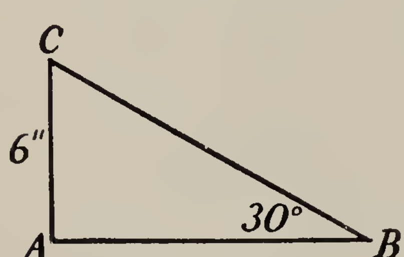


Fig. 126

11. In the right triangle ABC (Fig. 126) the side opposite the 30-degree angle is 6 inches long. What is the length of the side opposite the 90-degree angle?

67. Isosceles triangle. Equilateral triangle. A triangle having *two* equal sides is an **isosceles** triangle. A triangle having *three* equal sides is **equilateral.**

Parallel Lines

68. Meaning of parallel lines. Draw a line-segment AB (Fig. 127) about 12 cm. long. Place the sharp points of your compass on the ruler, or squared paper, 8 cm. apart. On AB lay off segment CD equal to 8 cm. in length.

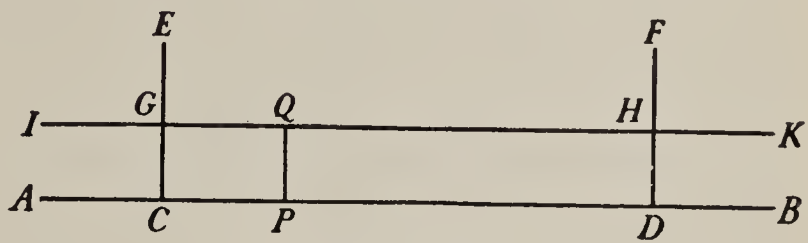


Fig. 127

At C and D draw lines CE and DF perpendicular to AB, using the protractor.

On CE and DF lay off two equal lengths, as CG and DH.

Draw line IK passing through G and H.

Line IK is said to be *parallel* to AB.

Measure angles DHK and CGI and show that CG and DH are perpendicular to IK.

Select any point P on AB. Draw a line perpendicular to AB at P and denote the point where it intersects IK by Q. Measure PQ.

How does the length of PQ compare with that of CG; of DH?

Because the perpendicular PQ which was drawn at a point P selected *anywhere* on AB, or its extension, is equal to the fixed lengths CG or DH, the lines AB and IK are said to be *everywhere* equally far apart. Hence, *they cannot meet however far they are extended.*

If two lines are drawn in the same plane surface, and if they do not intersect, however far extended, they are

called **parallel lines.** The word *parallel* means *running alongside of each other.*

By the *distance* between two *parallel lines* is meant the length of the perpendicular between them.

One of the properties of parallel lines is that they are everywhere equally distant.

69. Symbol for parallelism. The statement AB *is parallel to* CD is written briefly in symbols: $AB \parallel CD$.

Exercises

1. Point out parallel lines in the class room.

2. On a cube, or on a rectangular block, point out parallel lines.

3. Point out parallel lines on squared paper, on the ruler.

4. On a cube point out two lines which do not meet and are not parallel.

5. In the class room point out two lines which do not meet and are not parallel.

6. Are the rails in the adjoining picture parallel? Give reason for your answer.

Do they look parallel to you? Give reason for your answer.

7. Considerable knowledge of parallel lines was developed by the primitive races in connection with the art of weaving, basketry, and pottery. Many of the decorative designs are based on parallel lines, taking the shapes of rectangles, parallelograms, and squares. Try to collect designs in weaving, clothing, pottery, household implements, etc., which illustrate the use of parallel lines.

70. Drawing parallel lines. On squared paper draw two parallel lines AB and CD (Fig. 128) and a line EF intersecting AB and CD.

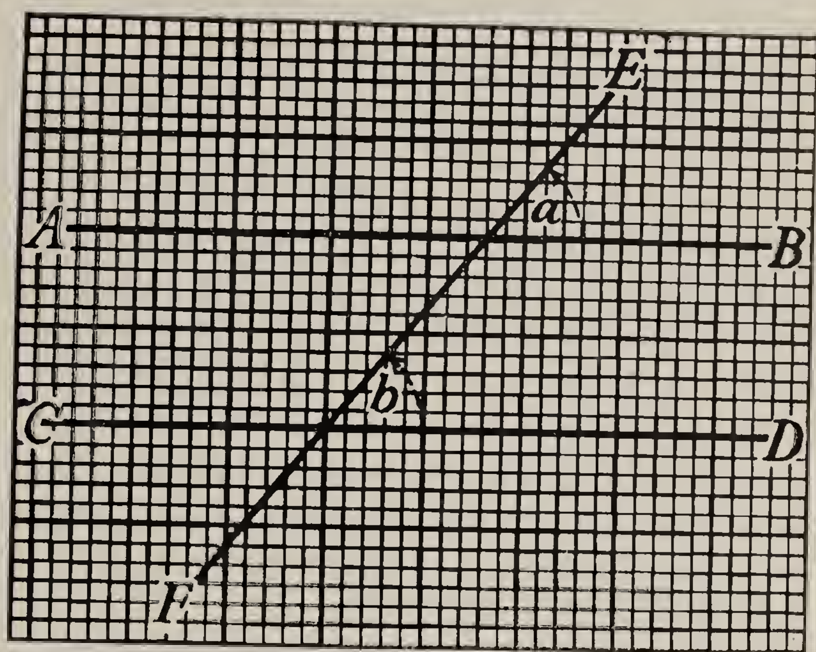


Fig. 128

Measure angles a and b.

How do they compare as to size?

The equality of angles a and b (Fig. 128) suggests the following methods of drawing parallel lines.

1. *The triangle method.* Place one side, AB, of the triangle ABC (Fig. 129) along EF. Draw a line along the side BC.

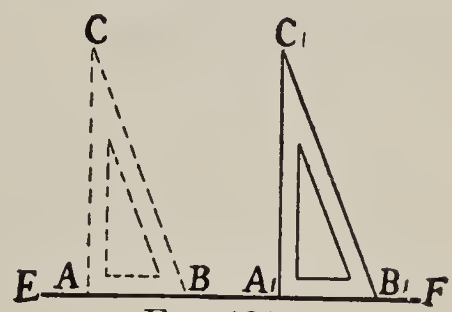


Fig. 129

Move the triangle by sliding side AB along EF until it takes the position $A_1\ B_1\ C_1$. Draw a line along $B_1\ C_1$.

Then BC is parallel to $B_1\ C_1$ because the angles at B and B_1 are equal.

2. *The T-square method.* Place the head of the T-square (Fig. 130) along one edge AB of the drawing board. Draw a line along the straight edge. Then move the head of the square downward

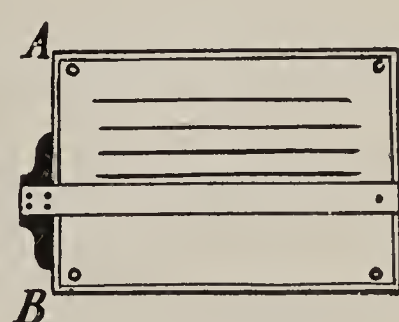


Fig. 130

along AB to the position of the desired parallel line, and draw a second line along the straight edge. Why are these lines parallel?

3. *The carpenter's square method.* Move one side of the square along an edge of the board (Fig. 131) and draw lines along the other side.

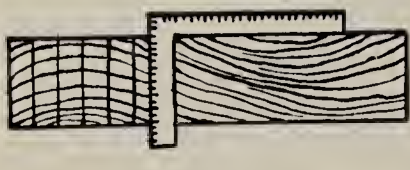


Fig. 131

4. *The protractor method.* Draw line CE (Fig. 132) and measure angle DCE.

At D draw angle BDF equal to angle DCE.

Then CE is parallel to DF. Why?

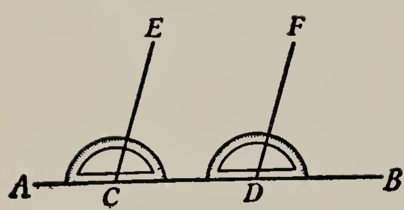


Fig. 132

5. *The parallel ruler method.* Draftsmen and navigators use a *parallel ruler* (Fig. 133) for drawing parallel lines. Two rulers are connected with cross pieces of brass, which work on pivots in such a way that the rulers may be spread apart or brought together, always remaining parallel to each other. This instrument is usually made of ebony.

Fig. 133

Exercises

1. A **parallelogram** is a quadrilateral whose opposite sides are parallel. On unruled paper draw a parallelogram whose adjacent sides are 4 cm. and 6 cm., respectively, including an angle of 40°.

Suggestion: Make the opposite sides parallel by using the protractor method (§70).

Measure the four sides. How do they compare as to size?

2. A **rectangle** is a parallelogram whose adjacent sides are at right angles to each other. Draw a rectangle having two adjacent sides equal to 3 cm. and 5 cm., respectively.

3. A **square** is an equilateral rectangle.
Draw a square whose side is 5 centimeters.

71. Angle pairs formed by three lines. If two lines AB and CD (Fig. 134) are intersected by a third line, eight angles are formed. These may be grouped in pairs as follows:

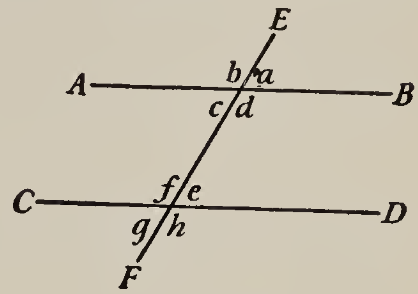


Fig. 134

Opposite angles: $a, c; b, d; e, g; f, h$.

Adjacent supplementary angles: $a, b; b, c; c, d; d, a; e, f; f, g; g, h; h, e$.

Corresponding angles: $a, e; b, f; c, g; d, h$.

Interior angles on the same side of EF: $c, f; d, e$.

Alternate interior angles: $c, e; d, f$.

Alternate exterior angles: $a, g; b, h$.

Exercises

1. It has been shown (§63) that opposite angles are equal. State the equations for angles
a and c, b and d, e and g, f and h.

2. State the equations for the adjacent angles
a and b, b and c, c and d.

3. If AB is parallel to CD, state the relations between a and e, d and h, b and f, c and g (§70). This may be stated as a geometric principle as follows:

If two parallel lines are cut by a transversal (cutting line), *the corresponding angles are equal.*

4. The methods of drawing parallel lines (§70) are based upon the principle that *two lines are parallel if the corresponding angles formed with a transversal are equal.*

We shall now *prove* that the *alternate interior* angles are equal if the *corresponding* angles are equal.

Let $a=e$.

We know that $a=c$. Why?

Hence we can replace in the first equation the number a by its equal c. This gives $c=e$.

5. Prove that $f=d$ when $b=f$.

6. Prove that $d=h$ if $f=d$.

7. Prove that *if the alternate interior angles formed by two lines and a transversal are equal the lines are parallel.*

Proof: If the alternate interior angles are equal, the corresponding angles are equal.

If the corresponding angles are equal the lines are parallel.

∴ If the alternate interior angles are equal the lines are parallel.

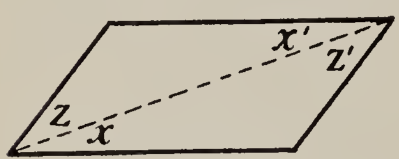


Fig. 135

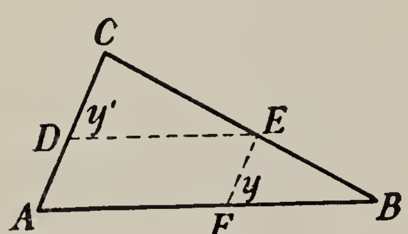


Fig. 136

8. One of two alternate interior angles formed by two parallel lines and a transversal is 66 degrees. The other is denoted by $11x$. What is the value of x?

9. Prove that in the parallelogram $ABCD$ (Fig. 135) the angles x and x' are equal. Prove that z and z' are equal.

10. Line DE (Fig. 136) is parallel to AB, and EF is parallel to CA. Prove that $y=y'$.

72. What every pupil should know and be able to do. The pupil should understand the meaning of the following terms: angle, vertex, sides; right, acute, obtuse, straight angle; adjacent, complementary, supplementary angles; perpendicular, parallel lines; isosceles, equilateral triangle.

Every pupil should know how to do the following:

1. To use the protractor to measure and draw angles; to draw perpendicular and parallel lines.

2. To solve equations of the forms $x+y=90$, and $6x+x+3x=180$.

3. To solve problems leading to equations of the form given in 2.

The following principles should be known:

1. *The sum of the angles of a triangle is 180°.*

2. *If two lines intersect, the opposite angles are equal.*

3. *The acute angles of a right triangle are complementary.*

4. *If two angles of a triangle are equal, the sides opposite them are equal.*

5. *If three angles of a triangle are equal, the triangle is equilateral.*

6. *In a right triangle with acute angles equal to 30 degrees and 60 degrees, the hypotenuse is twice as long as the side opposite the 30-degree angle.*

7. *If two parallel lines are cut by a transversal the corresponding angles are equal; the alternate interior angles are equal.*

The pupil should know the table of angular measurement, and be able to change degrees to minutes and seconds.

73. Typical problems and exercises. Every pupil should be able to answer questions and solve problems of the types given below.

1. Classify the following angles (Fig. 137).

FIG. 137

2. If the first angle of a triangle is twice as large as the second, and if the third angle is 6 times as large as the second, how large is each angle?

3. Draw a line perpendicular to a given line and passing through a point not on the given line.

4. Draw a line parallel to a given line and passing through a point not on the given line.

5. Change 18° 14′ 12″ to seconds.

6. One of two supplementary angles is twice as large as the other. How large is each?

7. Three angles just cover the plane around a point. They are denoted by x, $2x$, and $6x$. Find the number of degrees in each.

8. Draw a triangle two of whose angles are 40 degrees and 80 degrees, respectively.

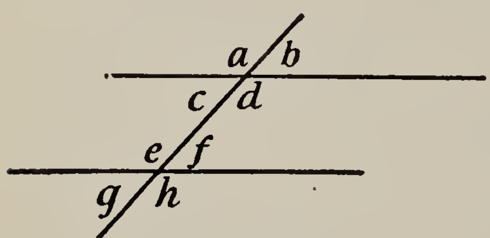


Fig. 138

9. State the relations for the angle pairs in Fig. 138.

10. Write a paper on one of the following topics:

a. Uses of angles in designs; in building; in navigation; etc.

b. Historical development of the angular unit.

c. Parallel lines.

CHAPTER V

USES OF LINE SEGMENTS AND ANGLES IN FINDING UNKNOWN DISTANCES

THE CONGRUENT-TRIANGLE METHOD OF FINDING DISTANCES

74. How to measure line-segments indirectly. We have been studying line-segments and angles. We are now prepared to study figures formed by more than two lines. The simplest of these is the triangle.

The triangle is used in ornamental work (Figs. 139, 140); in designing (Fig. 141); in construction (Fig. 142); in surveying (Fig. 143); in navigation (Fig. 144).

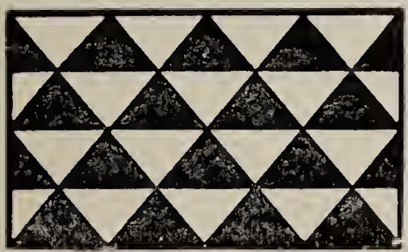


FIG. 139. TILE FLOORING

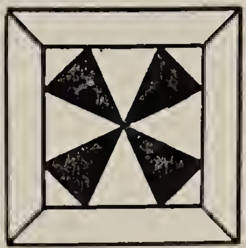


FIG. 140. PARQUET FLOORING

FIG. 141. DESIGN FOR A CHURCH WINDOW

FIG. 142. USE OF A TRIANGLE IN THE CONSTRUCTION OF A BRIDGE

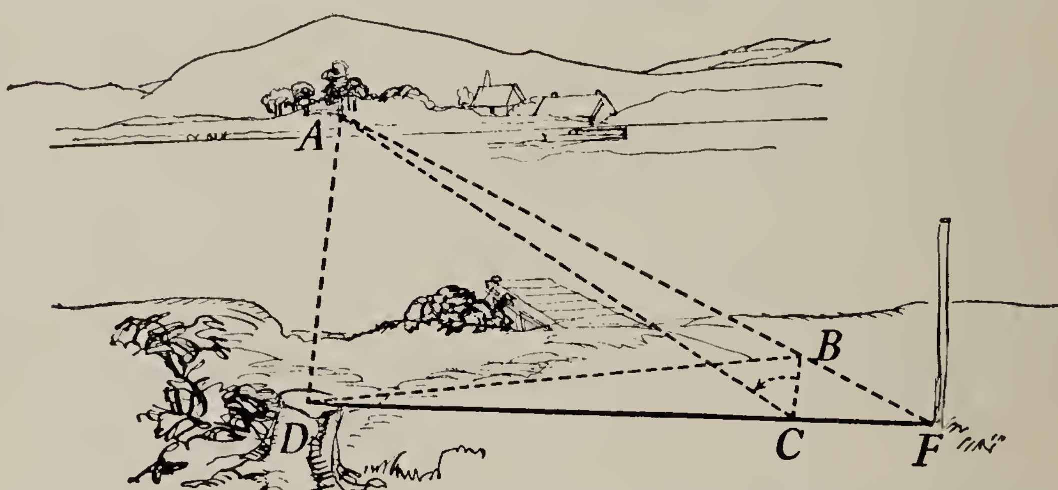


FIG. 143. SURVEYOR'S CHART FOR FINDING THE DISTANCE ACROSS A STREAM, *AD*

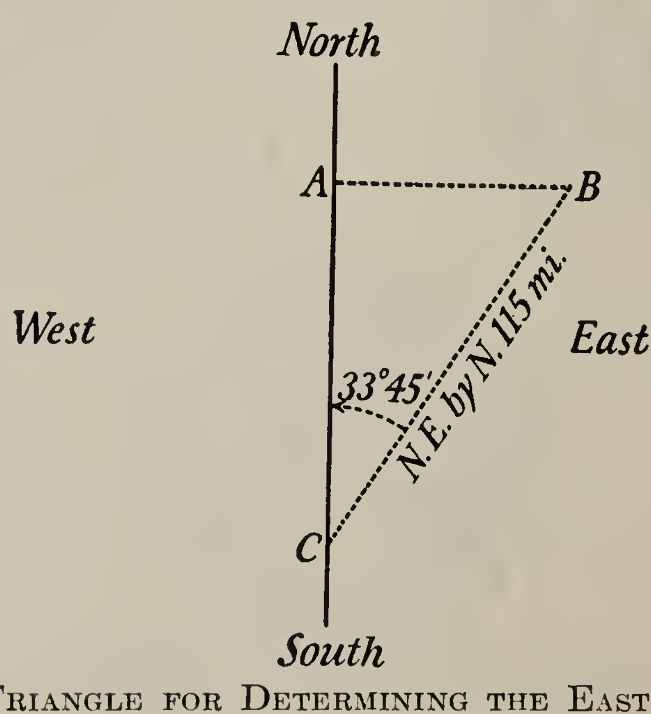


FIG. 144. TRIANGLE FOR DETERMINING THE EAST AND NORTH DISTANCES MADE BY A SHIP

In this chapter we shall learn to use the triangle to determine distances which cannot conveniently be measured directly, *e.g.*, the height of a tree (Fig. 145), the distance across a river (Fig. 146); or those which cannot be measured directly at all, as the distance through a building or a hill (Figs. 147, 148).

FIG. 145

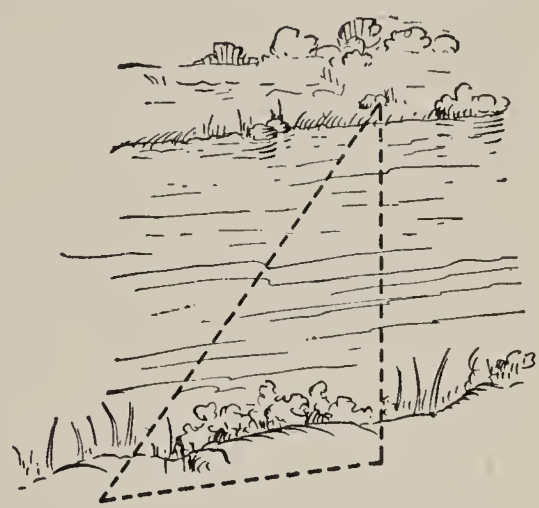


FIG. 146

FIG. 147

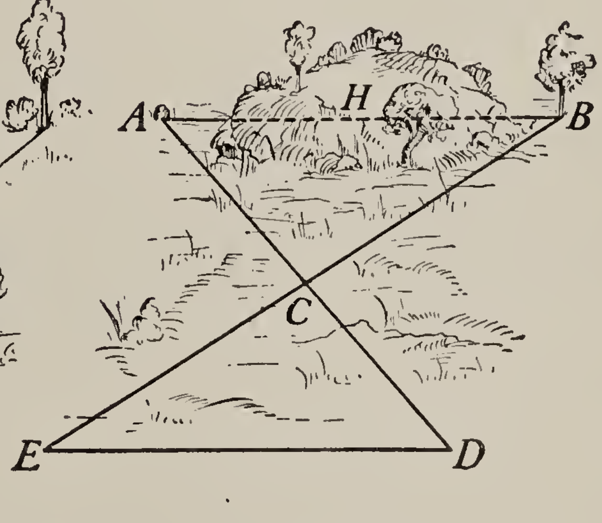


FIG. 148

To be sure, the height of a tree can be determined by climbing the tree and then applying a tape line. The distance across a river can be found by swimming across and measuring a line stretched from one bank to the other. A knowledge of mathematics, however,

makes this unnecessary. It saves work and time. Furthermore, by mathematics we can solve the problem even when direct measurement is impossible, as in Figures 147 and 148.

For example, in Fig. 148, to find the distance AB passing through the obstructing hill H, a point C is selected so that lines BE and AD may be laid off conveniently. A triangle ECD of the same size and shape as triangle ACB is laid off by making $CE = CB$, and $CD = CA$. The length of AB is then the same as that of ED, and can therefore be found by measuring ED.

Another method of determining AB makes use of a triangle of the same *shape* as ABC, but not of the same *size*. A third method uses a *right triangle*. All of these methods obtain the required length *without measuring the unknown line directly*. Determining distances without measuring directly is called **indirect measurement.**

75. Triangles of the same size and shape. In the problem of finding unknown distances it is necessary to know how to make a triangle which is exactly of the same size and shape as another triangle. One triangle is then an exact reproduction of the other. The two triangles are really the same triangle in two different positions, and one can be made to fit exactly on the other. Such triangles are called **congruent** triangles. This word comes from the Latin word *congruere*, meaning *to agree*. Congruent triangles have the sides and angles of one, equal to the corresponding sides and angles of the other. The symbol for congruence is $\cong$, the symbol $=$ meaning equal in size, and $\backsim$ meaning similar in shape.

Examples of congruence are the "blue prints" of the draftsman, the reprints of an original, the "negative" plate in photography.

Exercises

1. The story is told that a soldier of Napoleon was commanded by him to determine the width of a river which the army had to cross. In a brief time he brought in the desired information. When asked how he had obtained his result he said that he did it by means of mathematics as follows: Standing at the point S (Fig. 149) and

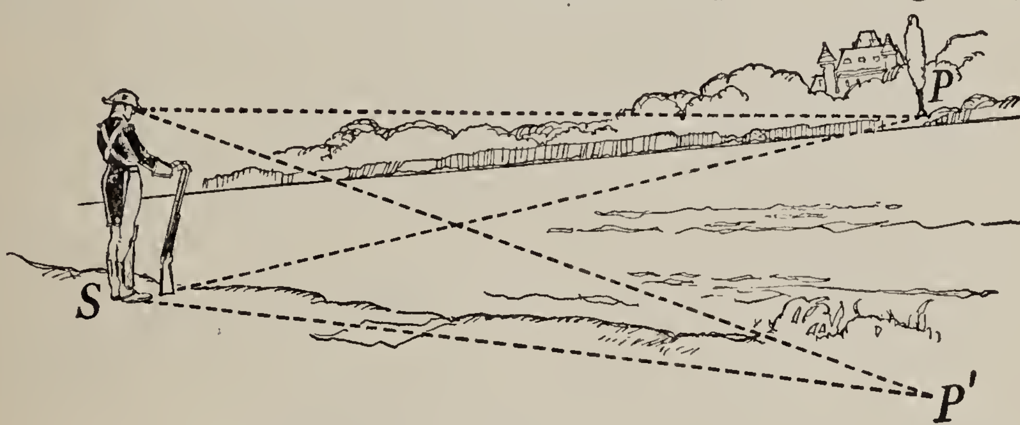


Fig. 149

looking across the river, he lowered his head until a point P on the opposite bank was exactly in line with the rim of his hat and his eye. Keeping his head in rigid position, he turned, sighted along the shore line, and had a stake placed at P', a point on the shore, exactly in line with his eye and the rim of his hat.

SP' was then measured and the required distance SP determined. Explain why $SP' = SP$.

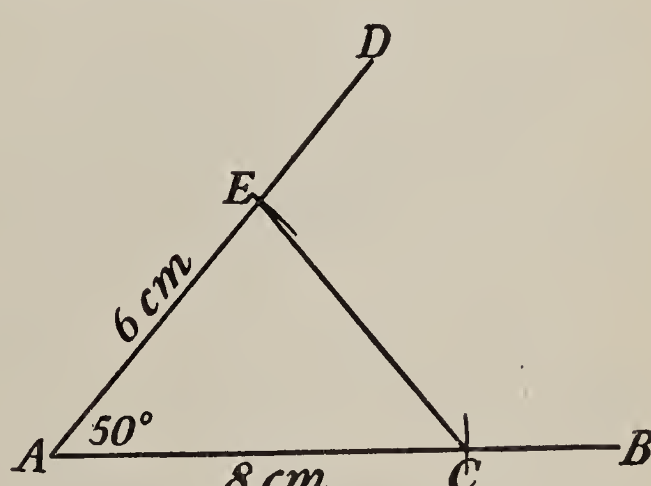


Fig. 150

2. Draw a line AB (Fig. 150) of indefinite length.

With the compass, lay off on AB a distance $AC = 8$ cm.

On AB at A draw an angle equal to 50 degrees.

On AD lay off $AE = 6$ cm.

Draw EC.

Cut triangle ACE from the paper, write your name on it, and pass it to the teacher.

By placing together all the triangles made by the pupils of the class, it is seen that they can be made to coincide. If all the drawings are exact, they will all fit exactly.

3. Draw a triangle ABC. Measure AB, AC, and angle CAB. As in Exercise 2, draw a second triangle with two sides equal to AB and AC, and with the angle included between these sides equal to angle CAB. Place the first triangle on the second. If your drawing is well-made, the two triangles can be made to fit exactly.

76. The congruent triangle method of finding unknown distances. Exercises 2 and 3 show that two triangles can be made to fit exactly if they have two sides of one equal to two sides of the other, and the included angles equal. These exercises illustrate a fact which is usually stated as follows:

Two triangles are **congruent** *if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*

Exercises

1. A surveyor may use the principle explained in §76 to find unknown distances as is shown in the following:

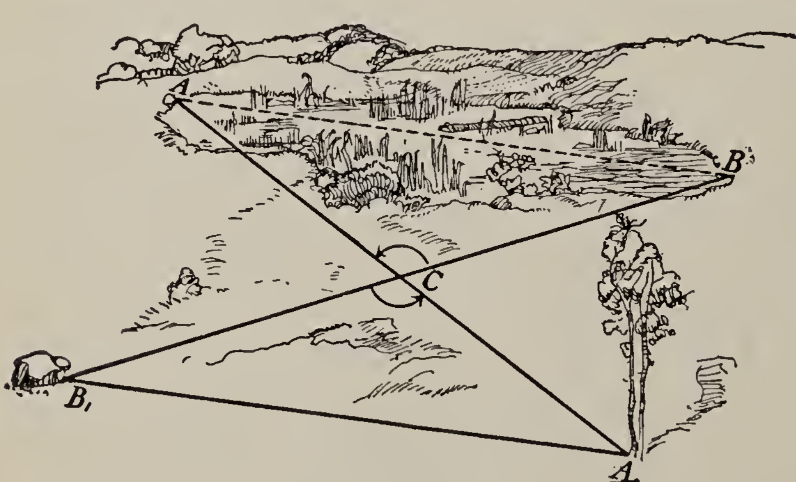


Fig. 151

Let AB (Fig. 151) be the unknown distance across a swamp, and let it be required to measure AB.

A suitable point C is selected so that lines may be drawn from A and B through C.

CB_1 is laid off equal to CB, and CA_1 equal to CA.

Then $A_1 B_1$ is drawn.

Show that triangle $ABC \cong$ triangle $A_1 B_1 C$.

Show that the length of AB may be found by measuring $A_1 B_1$, giving reasons for your statements.

2. Tell how to find an unknown distance by the congruent triangle method.

3. Draw a line AB (Fig. 152) of indefinite length.

Open the compass a distance of 8 cm. between the sharp points.

On AB lay off $AC = 8$ cm.

At A on AC draw an angle equal to 40 degrees.

At C draw an angle equal to 70 degrees.

Cut triangle ACD from the paper, write your name on it, and pass it to the teacher.

By placing the best drawn triangles together it will be seen that they coincide.

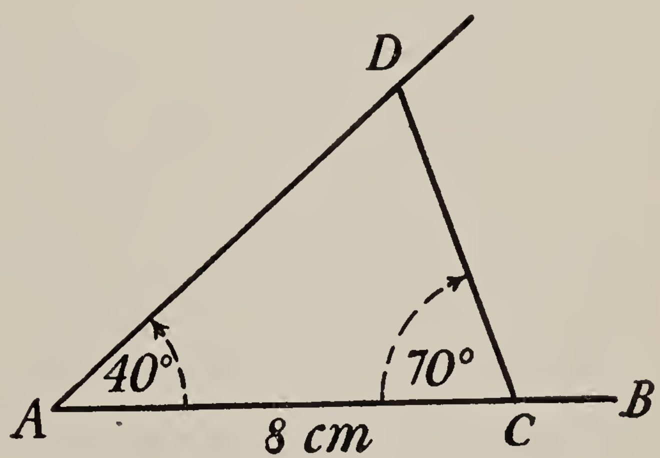


Fig. 152

Note that two angles and the side between their vertices in one triangle, are equal respectively to two angles and the side included between the vertices in the other triangle.

4. Draw a triangle ABC (Fig. 153). Draw a second triangle having two angles equal to angles A and B, and the side included between the vertices equal to AB.

Cut the second triangle from paper and see if you can fit it on $\triangle ABC$.

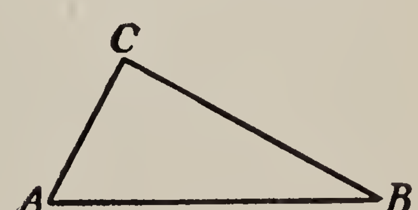


Fig. 153

Exercises 3 and 4 illustrate the following principle: *Two triangles are congruent*

if two angles and the side included between their vertices in one triangle are equal, respectively, to the corresponding parts of the other.

Exercise 5 shows how a surveyor may use this principle.

5. It is required to determine the distance from a point A on the shore line (Fig. 154) to a point C in the lake.

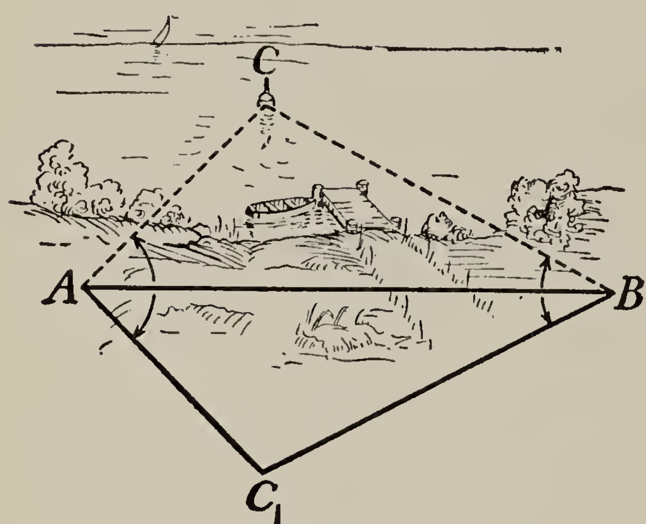


FIG. 154

A base line AB is laid off along the shore.

The transit is then placed at A and angle BAC is measured.

Line AC_1 is drawn, making angle BAC_1 = angle BAC.

Similarly, angle ABC_1 is laid off equal to angle ABC.

Show that triangle $ABC \cong$ triangle ABC_1.

Show that the length of AC may be found by measuring AC_1.

77. The nature and value of land surveying. References to surveying have been made repeatedly in the preceding pages. The art of surveying, *i.e.*, of measuring land, is probably as old as civilization. In Egypt, the science of surveying helped people to reestablish each year the boundaries of their land obliterated by the inundations of the Nile. The purpose of surveying is to establish certain boundaries of land and later to identify and locate these boundaries. The need of this is seen in our own history when the pioneer settlers followed Daniel Boone into the wilds of Kentucky and settled upon patches of land. They frequently marked their boundaries of land, or their *claims*, by chopping notches in trees. As time went on,

the crudeness of this method resulted in many disagreements concerning the ownership of particular patches and frequently in feuds.

On the other hand, when the Northwest Territory was opened to settlement about 1788 the Ohio Company of Associates, which purchased 1,000,000 acres of land, took great care to have the land properly subdivided and recorded and to have each settler's property clearly labeled and defined. For this purpose they

employed surveyors who traveled for miles through the wilderness, measuring the land and placing large stones for permanent land marks, the location of each being entered in the records. Later by congressional ordinance this work was extended by dividing the land into townships, the townships into sections and the sections into quarter sections.

Accuracy in surveying is exceedingly important. In a small town near Chicago, the surveyor in laying out village blocks used a surveyor's chain for measuring which was slightly too long. Later when each block was divided into lots there always remained a small piece of land in every block which did not belong to any lot. Frequent quarrels arise even to this day between claimants of this odd piece in each block.

Can you give examples from your reading of disagreements over staked claims in primitive settlements?

Do you know of disputes arising from peculiarities in land surveying?

What surveyor's marks have you seen in your neighborhood?

What American president was a skilled surveyor?

Find out what is meant by the Mason-Dixon line.

78. Instruments used in surveying. We have seen that the surveyor measures angles with the transit.

For measuring *distances* he uses chains, tapes, and rods. The *chain* (Fig. 155) consists of 100 links made of heavy steel wire. At every tenth link there is a brass tag, and at each end there is a handle. Some chains are 66 ft. long, the length of each link being

7.92 inches. Others have each link 1 ft. long making the length of the chain 100 feet. Chains are used when the territory is rough and extreme accuracy not essential.

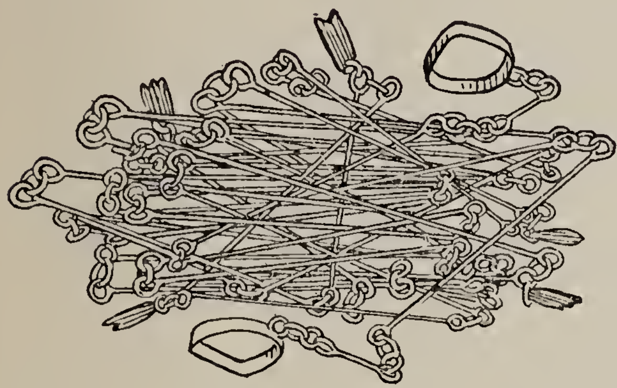


FIG. 155. SURVEYOR'S CHAIN

FIG. 156. STEEL TAPE

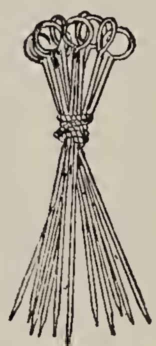


FIG. 157. CHAINING PINS

The *tape* (Fig. 156) is more convenient, less bulky, and more accurate than the chain. Tapes are usually 50 or 100 ft. long.

The *chaining pins* (Fig. 157) are used to mark points on the ground.

THE SCALE DRAWING METHOD OF FINDING UNKNOWN DISTANCES

79. Scale drawing. We have seen that distances which cannot be measured directly, such as the width of a river, or the height of a chimney, may sometimes be found by laying off a triangle congruent with a triangle having the required distance as one side. This is the *congruent triangle method of indirect measurement.* When it is impossible, or inconvenient, to lay off the triangle, other methods of finding the unknown distance are needed. The following example illustrates a method known as the *scale drawing method.*

Let it be required to find how far it is from one corner of the class room to the opposite corner.

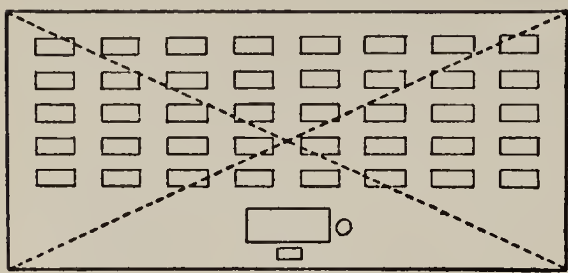


Fig. 158

Solution: Two consecutive sides of the room are measured and found to be 20 ft. and 30 ft. long, respectively (Fig. 158).

Let the side of a small square on squared paper represent 1 foot (Fig. 159).

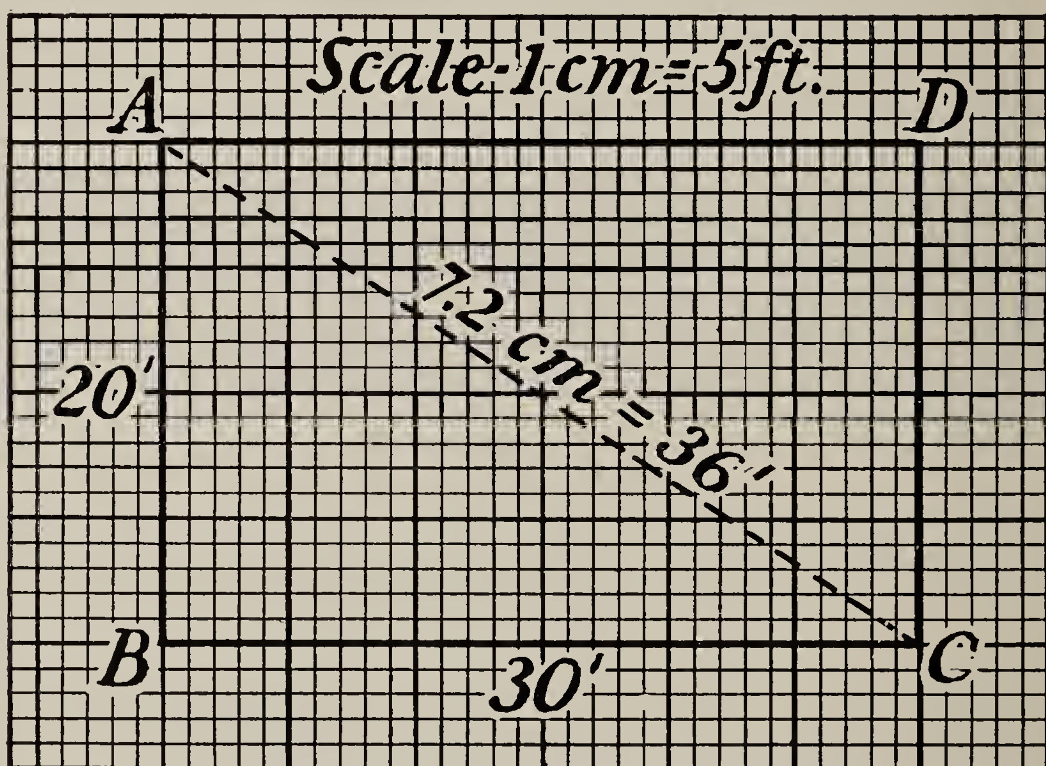


Fig. 159

Draw $BC=6$ cm, to represent the 30 ft. side.

Draw $BA=4$ cm, to represent the 20 ft. side.

Complete the rectangle $ABCD$.

Then $ABCD$ is a *scale drawing* of the class room floor with dimensions proportionally smaller than the *actual* dimensions. Thus, a length in the drawing equal

to a centimeter represents a length of 5 ft. of the floor. The drawing is then said to be made to the *scale:* 1 cm. = 5 ft.

The distance from A to C in the drawing is found by measurement to be about 7.2 cm., the 2 being estimated and therefore doubtful. The actual distance is (7.2×5) ft., or 36 feet approximately.

Summary: The preceding method of finding the distance AC, involves the following steps:

1. Lines and angles *related* to the required distance are measured. In the example above we measured AB, BC and $\angle B$ (Fig. 159).

2. A convenient scale is selected, measured lines are drawn to scale on squared paper, and the measured angles are drawn where they are needed to complete the figure.

3. The line representing the *required* distance, as AC is then drawn and measured.

4. The measurement obtained from the scale drawing is changed to the actual length.

Knowing how to draw to scale is important as it enables us to understand plans of land made by the surveyor, the maps used in geography, and the blue prints of the architect. The problem above shows that by measuring lines on the scale drawing we can determine the lengths of parts of the objects which the drawing represents, or the distances which cannot be measured directly. For this reason the *scale* must always be stated on the drawing.

80. Diagonal. A segment which joins two vertices of a polygon which do not lie in the same side is a **diagonal** of the polygon.

Exercises

1. Find the scale on a map of the state in which you live.

2. In the design of the hand mirror (Fig. 160) determine the *actual* lengths indicated by the dotted lines.

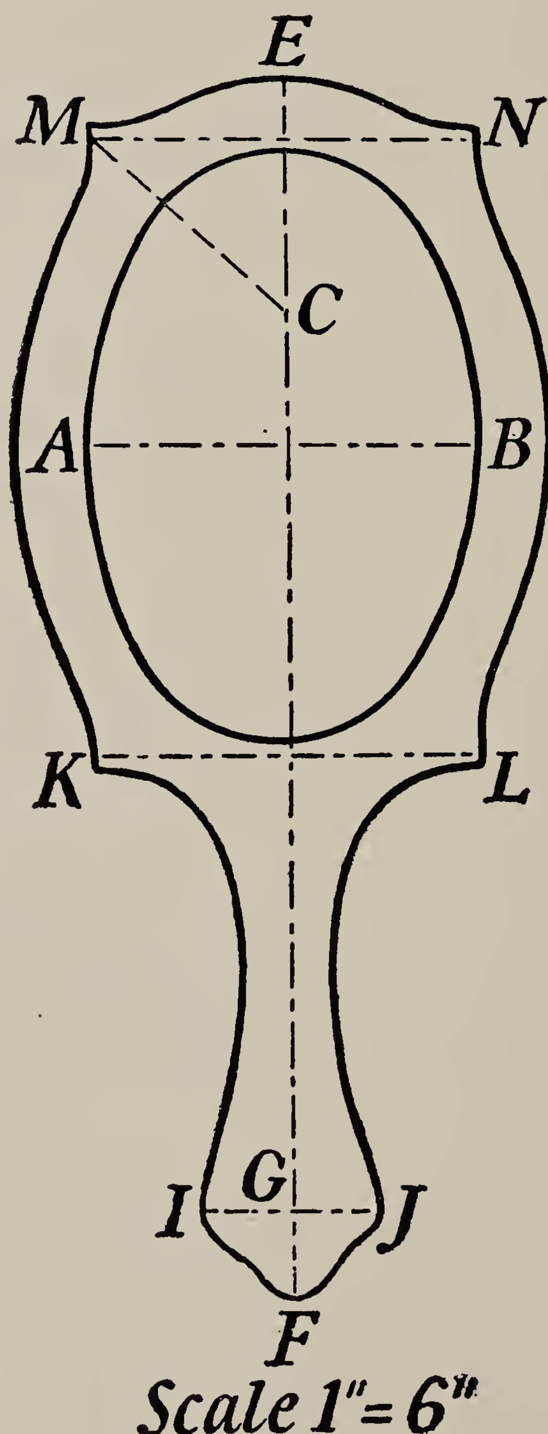


Fig. 160

3. Find the scale used in the diagram (Fig. 161).

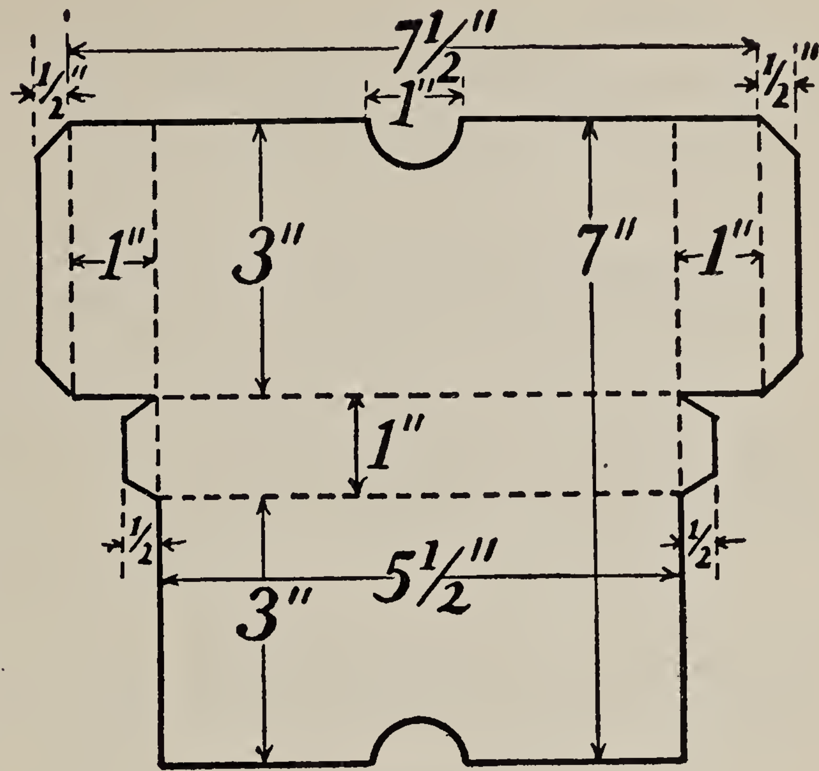
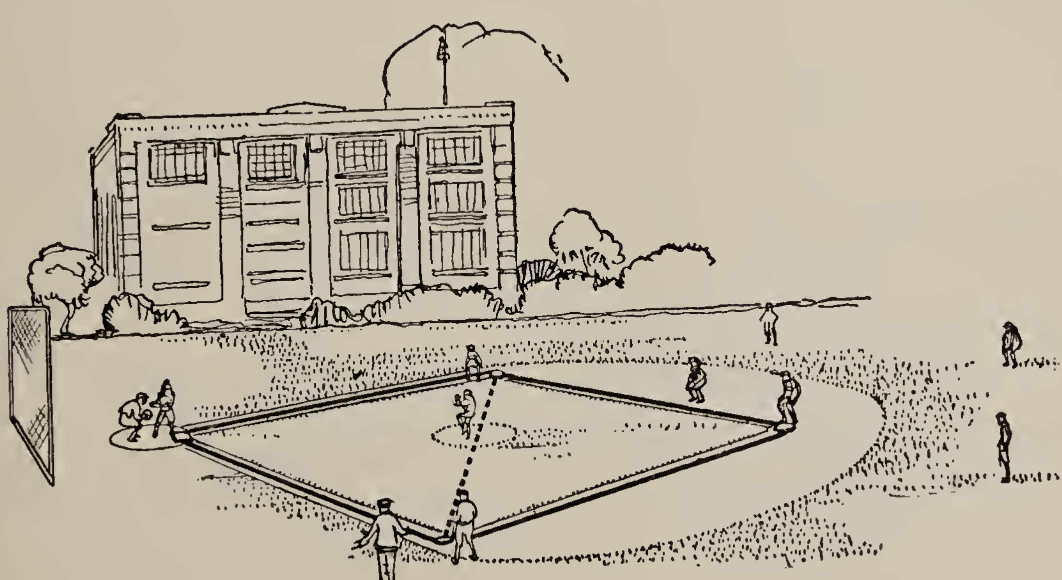


FIG. 161

4. Draw the design (Fig. 161) in actual size on a piece of heavy cardboard, cut along the solid lines, and then bend along the dotted lines. By joining the edges together by means of the flaps, a useful envelope case will be obtained. This may be mounted on a board $7\frac{1}{2}'' \times 3''$.

In working out the exercises below follow the directions given in §79.

5. A baseball diamond is of the form of a square whose side is 90 ft. long. Make a scale drawing of the diamond and find the direct distance of a throw from first to third base.



6. A man starting from a point P walks 60 yd. west and then 35 yd. north. What is his direct distance from P?

Suggestion: On a scale drawing to the right is *east,* and to the left is *west.*

7. In building a barn a carpenter aims to make the height of the roof (Fig. 162) equal to one-fourth of the width of the building. By means of a scale drawing find the angle between the rafter AC and the plate AB.

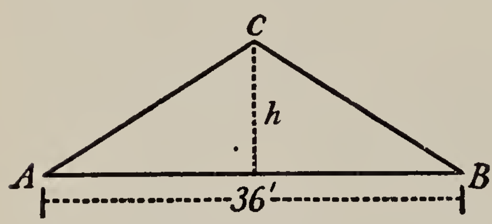


Fig. 162

8. Two automobiles leave a garage at the same time. One running at a rate of 20 miles an hour travels east for two hours, and then north for one hour. The other running at a rate of 25 miles an hour travels southwest for one hour and north for two hours. Find the distance between them at this time, using a scale drawing.

Suggestions: Make a rough sketch before attempting an accurate drawing. The required line is to be the *last* line drawn. Southwest means half way between south and west.

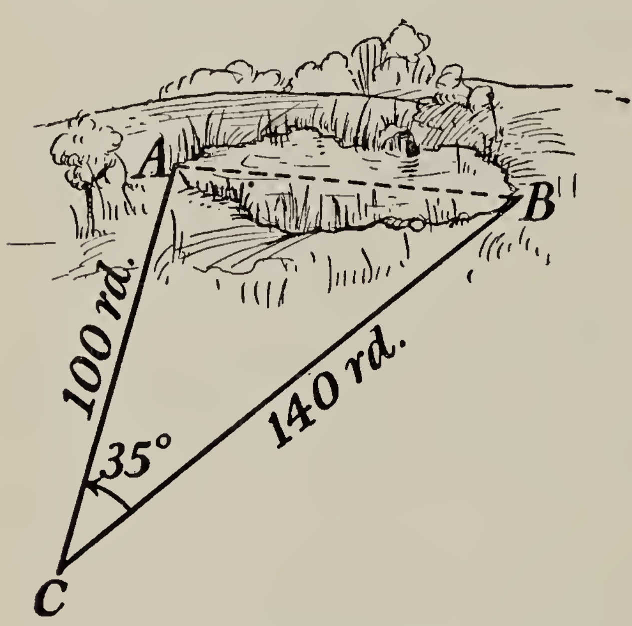


Fig. 163

9. Solve Exercise 1 (§76) by means of a scale drawing. The measured parts are given in Fig. 163.

Suggestion: Draw CA first.

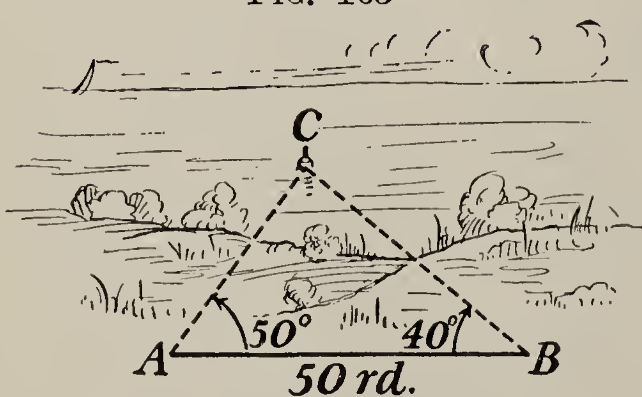


Fig. 164

10. Solve Exercise 5 (§76) by means of a scale drawing, the measured parts being shown in Fig. 164.

Suggestion: Draw AB first.

Check the accuracy of the drawing by finding the third angle, first from the formula $a+b+c=180$, and then by measuring it.

11. A surveyor wishes to find the distance AB (Fig. 165) between two points, A and B, on opposite sides of a river. B is so located that it cannot be seen from A.

He first draws a straight line DC through A.

He then lays off a distance of 160 ft. from A to C.

Placing his transit at C he measures angle ACB and finds it to be 45°.

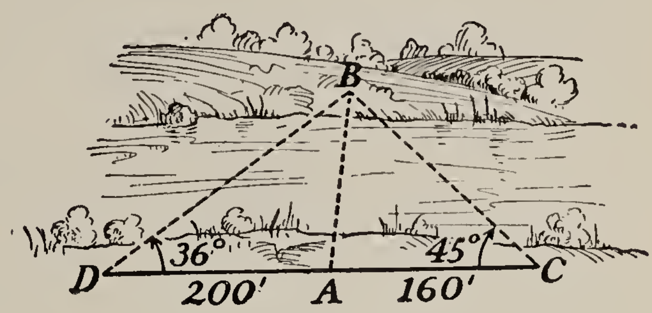


FIG. 165

Similarly, 200 ft. from A he locates D, and finds by measurement that angle $ADB=36°$. Find the distance from A to B.

81. Angle of elevation. To find the height of a chimney AB (Fig. 166) a surveyor places his transit at a point C taken at a convenient distance from A.

He then points the telescope horizontally in the direction ED.

Turning the telescope through angle DEB, he points it to the top of the chimney in the direction EB. Angle DEB is called the **angle of elevation** of the point B from the point E.

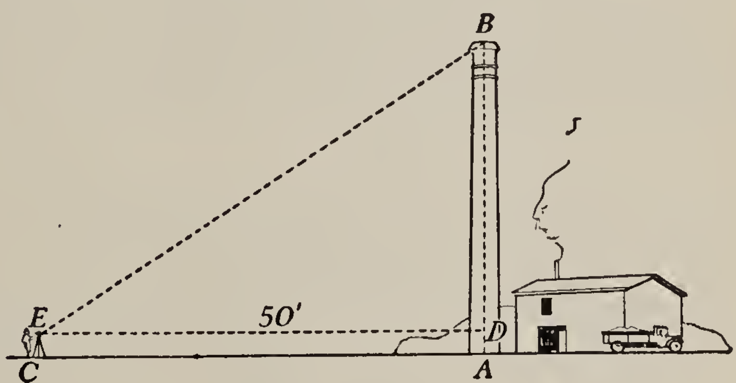


FIG. 166

EXERCISES

1. The telescope (Fig. 166) is 4 ft. above the ground and 50 ft. from A. Measure the angle of elevation, make a scale drawing of $\triangle EDB$, and from it find the length of DB. Find AB.

2. The angle of elevation of the top of a tree is 30° when observed at a point 40 ft. from the foot of the tree. How high is the tree above the horizontal line of sight?

To test the accuracy of the drawing use §66, Exercise 11, *i.e.*, measure the hypotenuse of the triangle and compare its length with the length of the side opposite to the 30° angle.

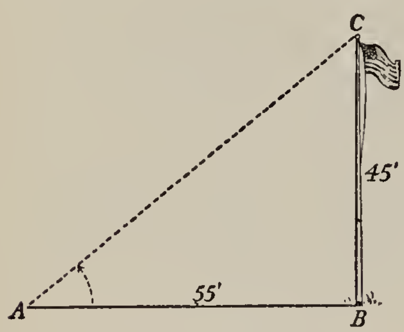


FIG. 167

3. A flagpole (Fig. 167) 45 ft. high casts a shadow 55 ft. long. Make a scale drawing and find $\angle CAB$. This is the angle of elevation of the sun.

4. When the angle of elevation of the sun is 30° a building casts a shadow 85 ft. long. Find the height of the building. Can you state a convenient way of checking the accuracy of your drawing?

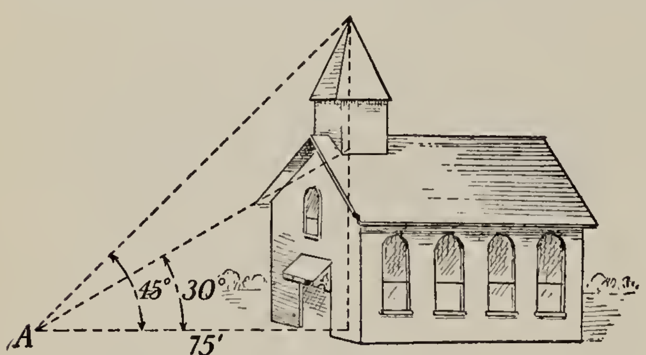


FIG. 168

5. On top of a building (Fig. 168) is a tower. At a point A, 75 ft. from the base of the building, the angle of elevation of the top of the tower is 45°, and the angle of elevation of the base of the tower is 30°. Find the height of the tower.

6. To determine the height of a tower AB (Fig. 169) a surveyor places his transit at a point C and finds the angle of elevation EC_1B to be equal to 68°.

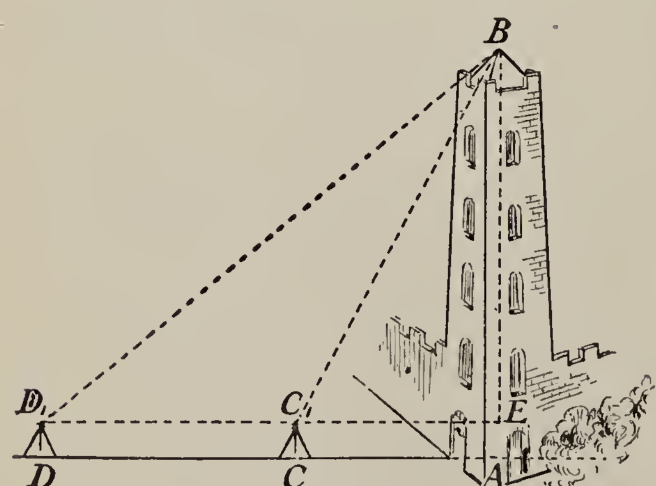


FIG. 169

He next places the transit at a point D in line with C and E and 60 ft. from C. He finds the angle of elevation ED_1B to be 35°. If in both cases the telescope was 3 ft. above the ground, find the length of AB.

82. Angle of depression. A transit is placed on top of a cliff A (Fig. 170) overlooking a river.

The telescope is first pointed *horizontally* in the direction AC.

It is then turned through angle CAB until it points

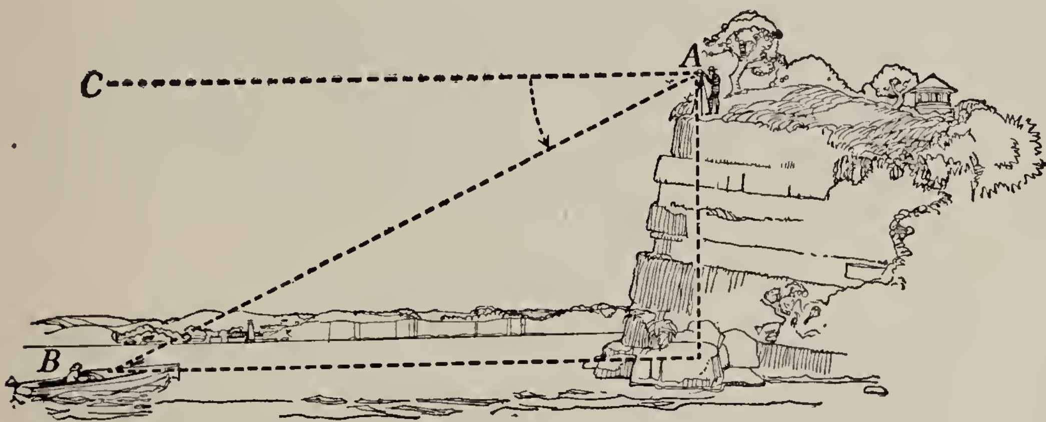


FIG. 170

to a passing boat B. The angle CAB is called the **angle of depression** of the boat from the point A.

Show that the *angle of depression* of B from the point A is the same as the *angle of elevation* of A from the point B.

EXERCISES

1. From the top of a lighthouse 100 ft. high (Fig. 171) the angle of depression of a boat is 50°. How far is the boat from the top of the lighthouse?

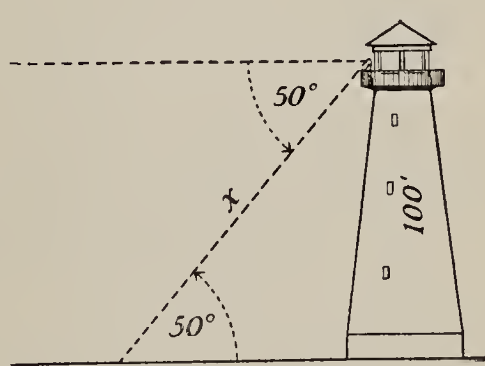


FIG. 171

2. An observation balloon B is anchored 2000 yd. above a point A. The balloonist observes the enemy at a point C and finds the angle of depression of C from B to be 62°. How far is it from A to C?

3. From the top of a cliff 100 ft. high the angle of depression of a buoy is 30°. Find the distance from the buoy to the bottom of the cliff.

The Similar Triangle-Method of Finding Unknown Distances

83. Advantages of the method. The use of the congruent triangle-method of finding distances is limited because it requires a triangle to be laid off which is congruent with the triangle containing the desired distance as a side. The scale drawing method is an improvement because the required triangle is drawn to scale on *paper*, not actually on the ground. However, the errors introduced in making a drawing, and the time spent in attempting to attain a high degree of accuracy in drawing and measuring, offer serious objections to this method. The "method of similar triangles," which is the next to be studied, has the advantage that an *exact* drawing is not needed, a rough sketch being sufficient. Furthermore, the final result is not determined by measurement, but by solving an equation, which makes greater accuracy possible.

84. Similar triangles. On squared paper draw a triangle, as ABC (Fig. 172).

Draw $A_1 B_1$ not equal to AB.

On segment A_1B_1 construct triangle $A_1B_1C_1$ so that angle $A =$ angle A_1, and angle $B =$ angle B_1.

Tell, without measuring, how angle C should compare with angle C_1.

Measure angles C and C_1 to test the accuracy of the drawing.

How do the two triangles compare as to shape? As to size?

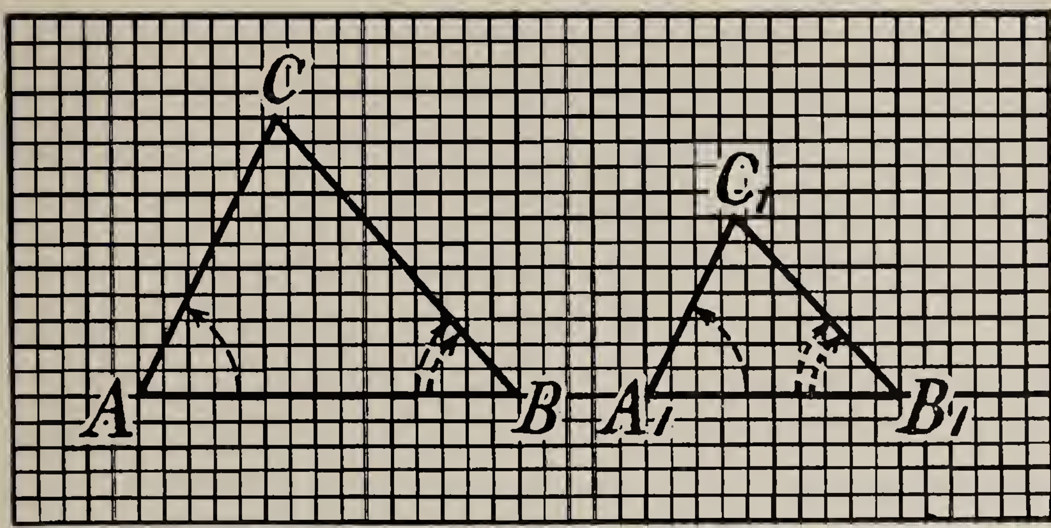


FIG. 172

Triangles having the same shape are *similar triangles.* Similar triangles do not have to be of the same *size.*

Using 2 cm. as a unit measure all the sides of the two triangles to two decimal places.

Compute to two decimal places the ratios

$$\frac{AC}{A_1C_1}=\frac{1.67}{1.23}=1.36, \text{ the 6 being doubtful.}$$

$$\frac{BC}{B_1C_1}=$$

$$\frac{BA}{B_1A_1}=$$

Computation:

$$\begin{array}{r} 1.36 \\ 1.23\overline{)1.67} \\ 1\ 23 \\ \hline 440 \\ 369 \\ \hline 71 \end{array}$$

If the measurements were accurate and the divisions exact, these ratios would be equal.

85. Similar polygons. The preceding exercise illustrates the following geometric facts:

1. *If the angles of one triangle are respectively equal to the angles of another, the two triangles are similar.*

2. *If the corresponding angles of two triangles are equal, the ratios of the corresponding sides are also equal.*

These two principles form the basis for the following definition of **similar polygons.**

3. *Two polygons are* **similar** *if the corresponding angles are equal and if the ratios of the corresponding sides are equal.*

Similarity has been shown in geometrical polygons. However, every figure drawn to scale is similar to the original. Maps and charts are similar in shape to the region which they represent. A photograph is similar to the object whose picture is taken. Knowledge of similarity is of importance to those who expect to become architects and designers, and to the engineers and machinists who have to use the plans drawn by the draftsmen.

86. Symbol for similarity. The symbol for similarity is $\backsim$. Thus, the statement *triangle ABC is similar to triangle* $A_1B_1C_1$ may be written briefly,

$$\triangle ABC \backsim \triangle A_1B_1C_1$$

Many objects are of the same shape and therefore similar, *e.g.*, any two squares, two equilateral triangles, a scale drawing and the figure it represents, the map of a piece of land and the land itself.

87. The similar triangle method. The following example illustrates the method of finding distances by means of similar triangles.

To measure the height, h, of a vertical pole, AC (Fig. 173) a boy measures the length of the shadow

AB and finds it to be about 76 ft. long. At the same time he finds that the shadow A_1B_1 of a 5 ft. vertical pole is 8 ft. long. What is the height of the first pole?

Solution: Each pole, its shadow, and the sun's ray passing over the top of the pole form a triangle.

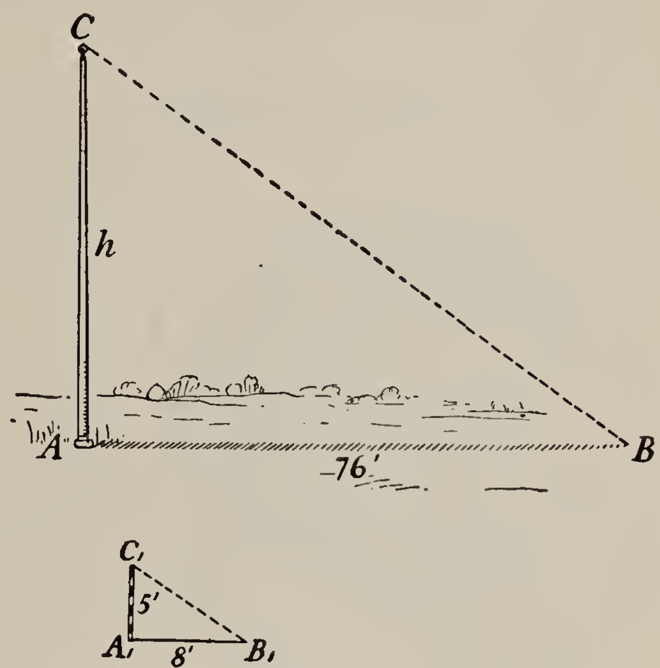


FIG. 173

The two triangles thus formed have two corresponding angles equal. For, angles A and A_1 are right angles, and angles B and B_1 are equal angles of elevation of the sun.

Hence $\triangle ABC \backsim \triangle A_1B_1C_1$. Why?

It follows that the ratios of the corresponding sides are equal, for example that

$$\frac{h}{5}=\frac{76}{8}.$$

We shall now learn how to solve this equation.

Multiplying both sides of the equation by 5, we have

$$\frac{\cancel{5}\cdot h}{\cancel{5}}=\frac{5\cdot \overset{19}{\cancel{76}}}{\underset{2}{\cancel{8}}}$$

By reducing the fractions it follows that

$$h=\frac{5\cdot 19}{2}=47\tfrac{1}{2}.$$

Hence the pole is about $47\frac{1}{2}$ ft. high.

The Greek mathematician Thales of Miletus (600 B.C.) is said to have used this method, known as the *shadow method,* to determine the height of the pyramids.

The adjoined picture of Thales expresses an artist's idea of the face of this famous ancient mathematician. You will find it interesting to know something about his life and work. He is known as the founder of the earliest Greek school of mathematics, but probably did not know as many mathematical facts as you do. In books on the history of mathematics you will find interesting stories about him.

THALES

88. Multiplication axiom. Equations of the type $\frac{h}{5}=\frac{76}{8}$ (§87) are met frequently in the solution of problems. Hence, we should understand how to solve them. In solving the equation, it was assumed that both members may be multiplied by the same number without destroying the equality. This principle of mathematics, known as the **multiplication axiom,** is usually stated as follows: *if equal numbers are multiplied by the same number the products are equal.* Since equations of the type above occur in the similar triangle method, Exercises 1 to 9 on page 131 are designed to give practice in the use of the multiplication axiom in the solution of equations.

Exercises

Solve the following equations:

1. $\frac{x}{18}=\frac{4}{9}.$

Solution: $\frac{x}{18}=\frac{4}{9}.$

$$\frac{\cancel{18}x}{\cancel{18}}=\frac{4\times\overset{2}{\cancel{18}}}{\cancel{9}},$$ by multiplying both members by 18. (Multiplication axiom)

$x=8,$ by reducing the fractions.

Check.

$$\frac{8}{18}=\frac{4}{9}$$

$$\frac{2}{9}=\frac{2}{9}.$$

2. $\frac{w}{25}=\frac{8}{15}.$

3. $\frac{l}{7}=\frac{3}{14}.$

4. $\frac{y}{10}=\frac{19}{6}.$

5. $\frac{a}{9}=\frac{5}{21}.$

6. $\frac{2}{3}=\frac{x}{24}.$

7. $\frac{m}{56}=\frac{75}{44}.$

8. $\frac{z}{8}=\frac{4}{12}.$

9. $\frac{7}{12}=\frac{x}{10}.$

Solve the following problems:

10. A boy scout, wishing to find the height of a pole (Fig. 174), holds a stick $\frac{1}{2}$ ft. long in vertical position, far enough from his eye so that it just covers the pole. If he stands 150 ft. from the pole, and if the stick is held 2 ft. from his eye, find the height of the pole.

Fig. 174

Suggestions: Make a sketch of Fig. 174. Locate two similar triangles in the drawing, and equate the ratios of the corresponding sides. Then solve the equation.

11. How high is a tree which casts a shadow 43 feet long, if at the same time a 7 ft. vertical post casts a shadow 9 ft. long?

Follow the suggestions given in Exercise 10 and find the height by solving an equation.

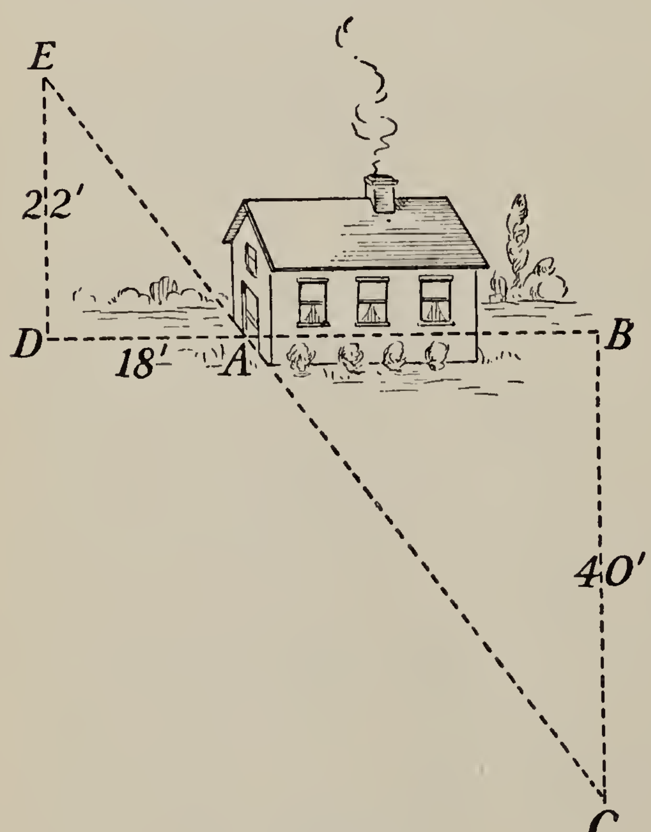


Fig. 175

12. To find the distance AB (Fig. 175) through a building, BC and DE were drawn at right angles to the line DAB. The following measurements were then made

$BC = 40$ ft.
$DE = 22$ ft.
$DA = 18$ ft.

What is the length of AB?

13. The sides of a triangle are 5 ft. 7 ft. and 8 feet. The longest side of a similar triangle is 15 feet. Find the other two sides.

14. The floors of two rooms in a new building are to be similar. The dimensions of one room are to be 15 ft. and 20 feet. The other room is to be 18 ft. long. Find the width.

15. The sides of a triangle are 2.9″, 5.2″, 6.3″. The shortest side of a similar triangle is 2.4″. Find the other two sides to two figures.

16. Find the height of your school building using the *shadow method* described in §87.

89. Proportion. A road bed AC (Fig. 176) rises to a height of 38 ft. for a horizontal distance AB of 110 feet. Let it be required to find how many feet the bed rises for a horizontal distance of 32 feet.

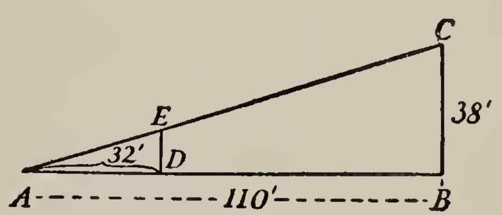


FIG. 176

This problem will be worked by two methods.

a. Solution by the scale drawing method. On squared paper draw to a convenient scale $AB = 110'$, $AD = 32'$, and $BC = 38'$. Draw $DE \perp AB$. Measure DE. This is the required length.

To check the accuracy of the result compute the ratios $\dfrac{DE}{38}$ and $\dfrac{32}{110}$ to three figures. Each of these ratios expresses the relative size of two distances. They should agree to two figures if careful work is done.

If two ratios, as $\dfrac{DE}{38}$ and $\dfrac{32}{110}$, are equal the statement $\dfrac{DE}{38} = \dfrac{32}{110}$ is called a *proportion*.

b. Solution by the similar triangle method. Show that triangles ABC and ADE (Fig. 176) are similar. Then it

follows that the ratios of the corresponding sides are equal, *i.e.*, $\frac{DE}{38}=\frac{32}{110}$.

Solve this equation to determine DE.

An equation of two equal ratios is a **proportion**.

Thus, $\frac{4}{6}=\frac{8}{12}$ is a proportion. Why?

Form three proportions.

Is the statement $\frac{6}{10}=\frac{3}{4}$ a proportion?

A proportion, as $\frac{2}{3}=\frac{20}{30}$, is read "2 over 3 is equal to 20 over 30," or "2 is to 3 as 20 is to 30."

Read the proportions

$$\frac{4}{16}=\frac{1}{4};\ \frac{1}{2}=\frac{8}{16};\ \frac{9}{10}=\frac{18}{20};\ \frac{a}{b}=\frac{c}{d}.$$

The equation $\frac{DE}{38}=\frac{32}{110}$ expresses a relation between horizontal distances and the rise of a road. It means that the ratio of one horizontal distance to another is equal to the ratio of the rise corresponding to the first to the rise corresponding to the second horizontal distance.

This type of relationship is found frequently. The following are examples in which it occurs.

1. A train moving at uniform rate travels 280 miles in 8 hours. How far does it travel in 14 hours?

Here we have the ratio of the two distances equal to the ratio of the corresponding number of hours, *i.e.*,

$$\frac{280}{x}=\frac{8}{14}.$$

2. An alloy of silver and copper weighing 90 oz. contains 6 oz. of copper. How much copper is there in 30 ounces?

Show that $\frac{90}{30}=\frac{6}{x}$.

3. If a sum of \$935 yields an annual income of \$46.75, what will be the income on \$500?

Show that $\frac{935}{500}=\frac{46\frac{3}{4}}{x}$.

90. Historical note. We have seen that if 3 of the 4 terms of a proportion are known, as in $\frac{x}{5}=\frac{76}{8}$, the fourth can be found by solving the equation for x. Rules for finding the fourth number of a proportion when three are known have been given considerable emphasis in the study of mathematics because of their great usefulness in the solution of many types of problems. Since three numbers are required in the operation, the method of finding the fourth became known as the *Rule of Three.* It has also been named the *Golden Rule.* "*The Rule of Three* is the chiefest and most profitable and the most excellent rule of all arithmetike for which cause it is said philosophers did name it the golden rule."—Humphrey Baker (1562).

The theory of proportion is as old as Plato's time (427–347 B.C.). There has been considerable diversity among mathematicians as to the notation used in writing proportions.* The question: if 2 apples cost 8

* See Cajori's *History of Elementary Mathematics*, pp. 193–203.

cents, what will 7 cost? in the notation of the Italian writer Tartaglia (1506–1557) would be stated thus:

If apples 2 || cost cents 8 || what be the cost of apples 7?

The older English arithmeticians write this as follows:

Apples	Cents
2	8
7	28

In the seventeenth century the following was customary:

Apples	Cents	Apples
2 ——	8 ——	7

Oughtred (1574–1660) wrote it

2 ; 8 :: 7

Thomas Dilworth (1784) used the notation

2 ·· 8 :: 7 ·· 28.

The notation of Leibnitz (1646–1716)

2 : 8 = 7 : 28.

was brought into use in the United States during the first quarter of the nineteenth century. It is read: 2 is to 8 as 7 is to 28.

The rule of three has played an important rôle in problem solving in England, America and Germany, especially in commercial circles.

91. Extremes and means. The *first* and *last* terms of a proportion as $a : b = c : d$, are called the **extremes,** the *second* and *third*, the **means.** Likewise, in the proportion $\frac{a}{b} = \frac{c}{d}$ the terms b and c are the means,

and a and d are the extremes. Name the means and the extremes in the following proportions: $\frac{1}{2}=\frac{4}{8}$; $\frac{6}{5}=\frac{18}{15}$; $\frac{m}{n}=\frac{x}{y}$.

92. Fundamental property of proportion. Write a proportion, as $\frac{3}{7}=\frac{6}{14}$. Find the product of the means; the product of the extremes. How do these products compare? Form another proportion, and compare the products of the means with the product of the extremes.

The preceding examples illustrate the principle that *in a proportion the product of the means is equal to the product of the extremes.*

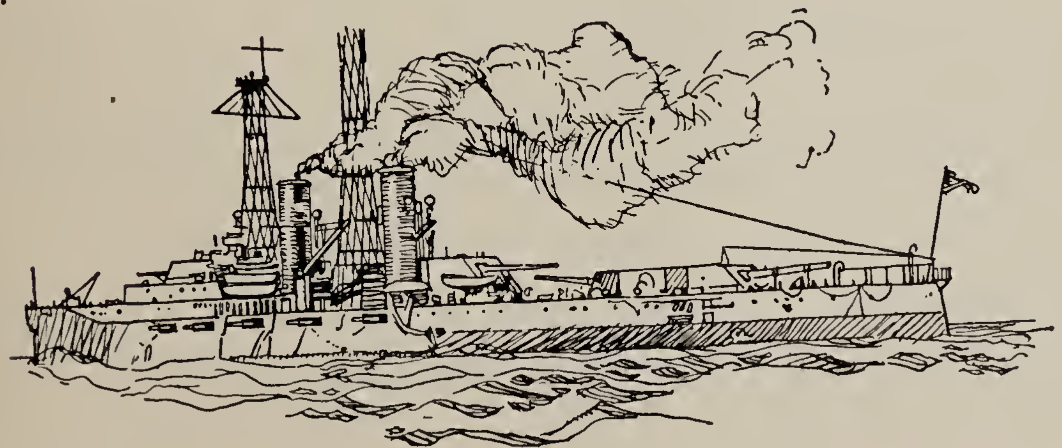
This principle suggests a convenient method of solving a proportion for the unknown number. This method is to be used in the exercises below.

Exercises

Solve the following problems by means of proportions:

1. A boat runs 30 mi. in 3 hours. How many miles will it run in 8 hours?

Solution: The ratio of the time of 3 hours to the time of 8 hours is $\frac{3}{8}$.



Denoting the required distance by x, the ratio of the corresponding distances is $\frac{30}{x}\cdot$

Since these ratios are the same, we have

$$\frac{3}{8}=\frac{30}{x}.$$

Since the product of the extremes, $3x$, is equal to the product of the means, 8×30, we have

$$3x=30\cdot8,$$

$$\therefore\ x=\frac{\overset{10}{\cancel{30}}\cdot8}{\cancel{3}}$$

$$\text{and}\qquad x=80.$$

2. If 12 acres of land yield 440 bushels of corn, at the same rate of yield how many acres would yield 200 bushels?

Suggestion: As in Exercise 1, state the proportion and then solve the equation.

3. A farm valued at \$11,400 is taxed for \$76.38. At the same rate what would be the tax on a farm valued at \$14,250?

4. A boy pays 90 cents for 2 doz. oranges. What is the price of 20 oranges?

5. Five bars of soap are sold for 35 cents. At the same rate, find the price of 8 bars.

6. If the simple interest on a sum of money for 6 years is \$200 what will be the interest for 10 years?

7. The dimensions of a rectangle are 9 inches and 5 inches. Find the width of a similar rectangle 12 inches long.

Suggestion: Make a sketch before writing the equation.

8. The shadows of a pole and a 5 ft. rod are respectively 80 ft. and 7 ft. Make a sketch and then find the height of the pole by means of an equation.

9. If a bushel of shelled corn weighs 56 lb., how many ounces does a pint weigh?

10. The food parts of beef are protein, fat, and water. In 5 lb. of sirloin steak there are 13 oz. protein, 14 oz. fat, and 42 oz. of water, the remainder being waste material. How many ounces of protein, fat, and water are there in 3 lb. of sirloin steak?

11. If $\frac{7}{8}$ yd. of lace cost 63 cents, how many yards can be purchased for $3.50?

12. If 3 yd. of ribbon cost $2.70, what is the price of $1\frac{3}{4}$ yards?

13. If an automobile runs 16 mi. on a gallon of gasoline, how much gasoline will be consumed on a 350 mile trip?

14. If a stenographer writes 475 words in 3 minutes, how long will it take her to write 2000 words?

15. If a $64.80 tax is paid on property assessed at $2575, what tax should be paid on property assessed at $6000?

16. If 230 lb. of milk produce 8.3 lb. of butter fat, how many pounds of milk will be required to produce 35 lb. of butter fat?

17. If 86 lb. of metal make 15 castings, how much metal will be required to make 10 castings?

18. If 18 yd. of silk cost $48.50, what will 35 yd. cost?

The Right Triangle Method of Finding Unknown Distances

93. Advantages of this method. Several methods have been used to determine unknown distances by indirect measurement. Each method is an improvement over the preceding methods. As we continue the study of mathematics, we also continue to improve our methods. In the method explained below the number of measurements required to solve the problem is

reduced to a minimum. This also reduces the number of errors. The result does not depend on the accuracy of a drawing. For this reason it is generally used in practical work, such as surveying, where accuracy is important. It is called the *right triangle method.* The method is made clear in §§94–96.

94. Similar right triangles. On squared paper draw a right triangle (Fig. 177) having one acute angle equal to 30°. Measure to two decimal places the side a, *opposite* the 30° angle, and the side b, *adjacent* to the 30° angle. Find the ratio $\frac{a}{b}$ by dividing a by b to two decimal places.

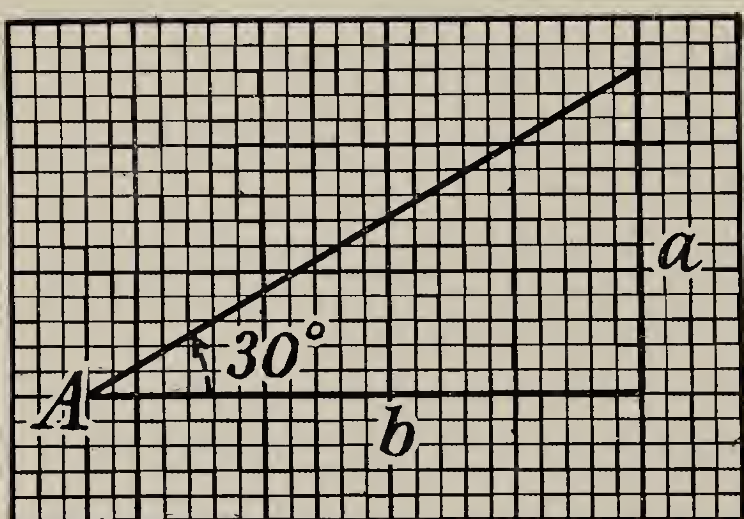


Fig. 177

The results found by the pupils in the class should agree. For according to §84 all these triangles, if constructed *exactly*, are similar to each other. They are therefore really the same triangle drawn to different scales.

95. Table of tangents. We have seen (§94) that for all right triangles having one acute angle equal to 30°, the ratio of the side *opposite* the 30° angle, to the side *adjacent* to the 30° angle is the *same.*

We may now draw other right triangles with acute angles of various sizes and make a table which states

for each acute angle the ratio of the side *opposite* to the side *adjacent*. Such a table is called a **table of tangents,** and the ratios are called **tangent ratios.** The

TABLE OF TANGENTS OF ANGLES FROM 0° TO 89°

Angle	*Tangent*	*Angle*	*Tangent*	*Angle*	*Tangent*
0	.000	30	.577	60	1.732
1	.017	31	.601	61	1.804
2	.035	32	.625	62	1.881
3	.052	33	.649	63	1.963
4	.070	34	.675	64	2.050
5	.087	35	.700	65	2.145
6	.105	36	.727	66	2.246
7	.123	37	.754	67	2.356
8	.141	38	.781	68	2.475
9	.158	39	.810	69	2.605
10	.176	40	.839	70	2.747
11	.194	41	.869	71	2.904
12	.213	42	.900	72	3.078
13	.231	43	.933	73	3.271
14	.249	44	.966	74	3.487
15	.268	45	1.000	75	3.732
16	.287	46	1.036	76	4.011
17	.306	47	1.072	77	4.331
18	.325	48	1.111	78	4.705
19	.344	49	1.150	79	5.145
20	.364	50	1.192	80	5.671
21	.384	51	1.235	81	6.314
22	.404	52	1.280	82	7.115
23	.424	53	1.327	83	8.144
24	.445	54	1.376	84	9.514
25	.466	55	1.428	85	11.430
26	.488	56	1.483	86	14.301
27	.510	57	1.540	87	19.081
28	.532	58	1.600	88	28.636
29	.554	59	1.664	89	57.290

table on page 141 gives the tangent ratios* for acute angles from 0° to 89°. Thus for an angle of 30° we find the tangent ratio to be .577. Briefly, we say *the tangent of 30° is .58,* which may be written

$$\tan 30° = .58 \text{ approximately}$$

Exercises

1. From the table of tangents find the tangent ratios of the following angles 10°; 22°; 45°; 67°; 82°.

In each case state your result in the form of an equation, as $\tan 10° = .176$.

2. Find the angles corresponding to the following tangent ratios: .141; .306; .601; 1.73; 6.31; $\frac{5}{4}$; $\frac{7}{8}$. State results in the form of equations.

96. The right triangle method. Exercise 1, below, shows how to find unknown distances by measuring two parts of a right triangle, and using the table of tangents. This is the **right triangle method.**

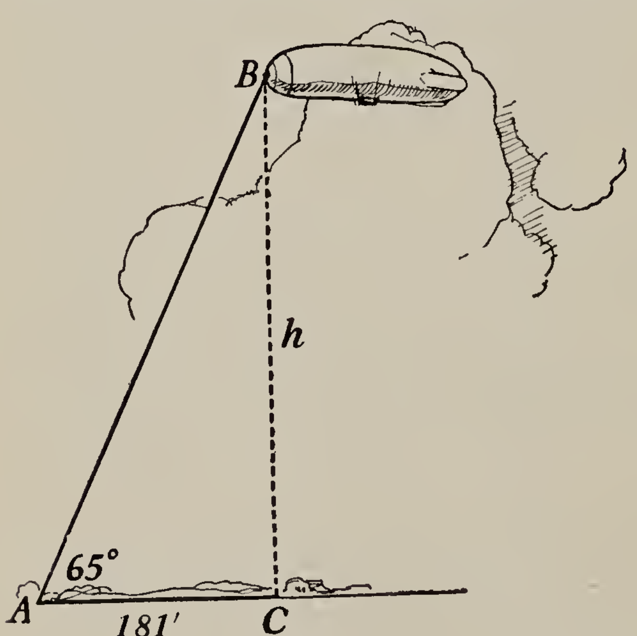


Fig. 178

Exercises

1. An observation balloon B (Fig. 178) fastened by a cable AB at A, is directly over a point C which is about 181 ft. from A. The angle of elevation of B at A is 65° approximately. How high is the balloon?

* There are other ratios, *e. g.*, the opposite side to the hypotenuse, and the adjacent side to the hypotenuse. The table of tangents is sufficient for the solution of the problems which follow.

Solution: $\frac{h}{181}=\tan 65°$. Why?

From the table, $\tan 65°=2.145$.

Hence, $\frac{h}{181}=2.145$.

Multiplying both members of the equation by 181,

$$h=(181)\ (2.145)=388.245.$$

Since in the product (181) (2.145) the 5 in the 2.145 and the 1 in 181 are uncertain, the first partial product 2145 is uncertain. Furthermore, in the other partial products, the 8×5 and 1×5 are uncertain. Hence, in the sum of the partial products the last four figures are uncertain.

Computation:

$$\begin{array}{rl}
2.14\dot{5}\times18\dot{1}, & \\
\hline
\dot{2}\dot{1}\dot{4}\dot{5} & =\text{first partial product,} \\
1716\dot{0} & =\text{second partial product} \\
214\dot{5} & =\text{third partial product} \\
\hline
38\dot{8}.\dot{2}\dot{4}\dot{5} & =\text{product.}
\end{array}$$

All the doubtful numbers in the computation above have been marked with dots placed over the number. Since the second 8 in 388 is doubtful, the figures following are meaningless, and should be dropped.

$\therefore$ The balloon is about 388 ft. high.

2. Make a summary of the steps in the solution of Exercise 1.

3. The rope of a flagpole is stretched so that it touches the ground at a point 18 ft. from the foot of the pole. The rope makes an angle of 70° with the ground. Find the approximate height of the pole, writing your computation as shown in Exercise 1.

4. A pole 22 ft. high casts a shadow 16 ft. long. Find the angle of elevation of the sun.

Solution: Let x be the required angle.

$$\tan x=\frac{22}{16}.$$

Since the last figures in 22 and 16 are obtained by measurement and are therefore doubtful, the division shows that in the quotient the first figure to the right of the decimal is doubtful and the second meaningless.

Hence $\tan x = 1.4$, and from the table

$$x = 54° \text{ approximately.}$$

Computation:

```
      1.37
16)22
   16
    6.0
    4.8
     120
     112
```

5. A vertical pole 9 ft. long casts a shadow, on level ground, 11 ft. long. Find the angle of elevation of the sun.

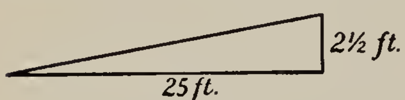


FIG. 179

6. What is the angle of elevation of a road which rises $2\frac{1}{2}$ ft. in a horizontal distance of 25 feet (Fig. 179)?

7. In the triangle (Fig. 180) find the approximate values of angle x for the following values of a and b.

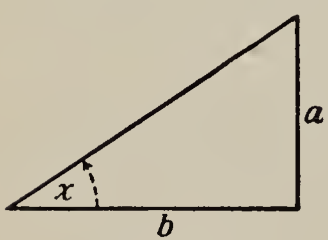


FIG. 180

a	231	18.4	1.57
b	172	15.3	1.83

8. Find the value of a (Fig. 180) corresponding to the value of x and b given in the following table.

b	165	28.3	204
x	20°	42°	35°

97. What every pupil should know and be able to do. Having studied Chapter V every pupil should be able to do the following.

1. To find unknown distances by the methods taught in the chapter.

2. To solve for the unknown equations of the type $\frac{x}{3}=\frac{16}{5}$.

3. To determine the degree of accuracy that can be obtained in the product of two decimal fractions in which the last figure to the right is doubtful.

4. To use compass, protractor, and squared paper in drawing triangles, angles, and measuring segments.

5. To solve some simple verbal problems by means of proportions.

The following principles should be known:

1. *Two triangles are congruent, if two sides and the included angle of one are equal to two sides and the included angle of the other.*

2. *Two triangles are congruent if two angles and the side included between their vertices in one triangle are equal, respectively, to the corresponding parts of the other.*

3. *If the angles of one triangle are equal to the angles of another, the triangles are similar.*

4. *If the corresponding angles of two triangles are equal, the triangles are similar, and the ratios of the corresponding sides are equal.*

5. *In a proportion the product of the means is equal to the product of the extremes.*

6. *If equal numbers are multiplied by the same, or equal, numbers the products are equal (multiplication axiom).*

7. The pupil should be familiar with the meaning of the following terms: congruent polygons, scale drawing, angle of elevation, angle of depression, similar polygons, proportion, tangent ratio.

98. Typical problems and exercises. The pupil should be able to solve the following problems and exercises:

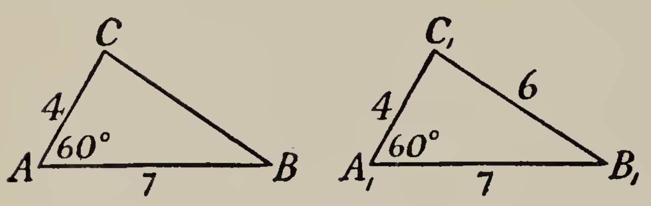


Fig. 181

1. In triangle ABC (Fig. 181) find the length of BC. Give reasons for your answer.

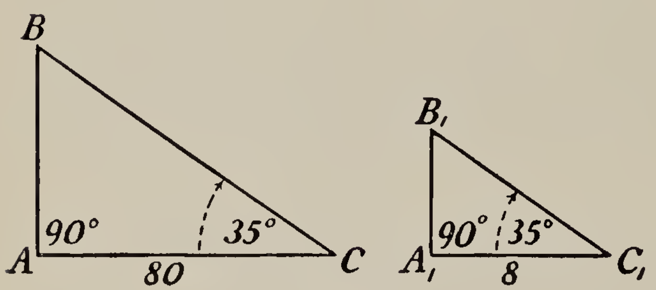


Fig. 182

2. In triangles ABC and $A_1B_1C_1$ (Fig. 182) find the lengths of AB and A_1B_1.

3. Draw a triangle. Draw a second triangle congruent with the first. Draw a third triangle similar to the first.

4. Find the distance AB (Fig. 183) by means of a scale drawing.

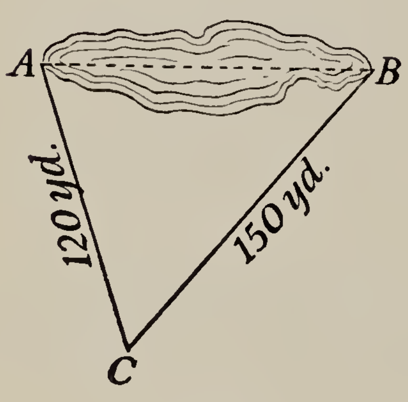


Fig. 183

5. Find the height of a building the shadow of which is 90 ft. long when the angle of elevation of the sun is 42 degrees. Use first the scale drawing method; then the right triangle method, tan 42 degrees being .900.

6. The height of the shadow of a flagpole is 72 ft., when the shadow of a 6 ft. vertical rod is 10 feet. Find the height by the similar triangle method.

Solve for x.

7. $\frac{x}{3}=16;$

8. $\frac{x}{10}=\frac{3}{5};$

9. $\frac{86}{x}=\frac{5}{8};$

10. $\frac{3.5}{6.3}=\frac{x}{.75}.$

11. By means of a proportion, find the tax on a property assessed at $18,000, if a tax of $72.40 is paid on property assessed at $3280.

12. Determine to three figures the ratio $\frac{8.72}{9.36}$.

13. Write a paper on one of the following topics:

a. Uses of the triangle in ornamental work; in construction; in designing; in surveying.

b. Indirect measurement.

c. Land surveying.

d. Proportion.

e. Various methods of finding unknown distances.

f. Importance of congruence, similarity.

g. The life and work of Thales.